The genericity assumption in electrical network theory

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1 Abstract

When trying to solve linear, time-invariant electrical networks, in many cases numerical methods are insufficient on their own and a more general, combinatorial approach is needed to check the singularity of the given network before they can be applied. This is especially true in case of large networks where the models are more likely to be inaccurate, and these singularities can lead to incorrect results.

In the past, several authors gave necessary and sufficient conditions of unique solvability, mainly using matroid theory which offers polynomial algorithms for checking these conditions. To obtain such conditions, some sort of generality of the parameters in the describing equations of the network had to be assumed, but these assumptions were formulated in different ways, leading to different, non-equivalent results.

The purpose of this paper is to highlight the role of the genericity assumption in these statements, and the difference between the weaker and the stronger cases. I would also like to further discuss its properties, the effect this assumption has on the system of describing equations, and how it can alter the rank of a matrix.

The last part, presented in Section 4, is a new result contained in a paper that we submitted for publication.

2 Multiports, and related subjects

To describe what the genericity assumption means in electrical network theory, the introduction of some concepts from the field of mathematics and electrical network theory is needed.

Throughout the paper we use standard matroid notation. $\oplus$ denotes the direct sum, * denotes the dual of a matroid, / denotes contraction and $\vee$ denotes the union of matroids. For further matrodial concepts, the reader is referred to [1] and [2].

2.1 Matrix description of a multiport

In electrical network theory, devices are often modelled as *n-port*s. An n-port – or *multiport* if the number n has no significance – is an abstract network element, with n pairs of terminals (see Figure 1) and k linearly independent equations describing the voltages and currents of these ports. These equations can be written in the form $\mathbf{Au}+\mathbf{Bi} = \mathbf{0}$ with $\mathbf{u}$ and $\mathbf{i}$ being vectors of height n, representing the voltages and currents of each port, respectively, and $\mathbf{A}$ and $\mathbf{B}$ being $k \times n$ real matrices. It is important to note, however, that while these matrices uniquely determine the n-port by describing the relationships between $\mathbf{u}$ and $\mathbf{i}$, the converse is not true, one n-port has several different matrix descriptions. The rank of an n-port can be defined as follows: $r = r(\mathbf{A}|\mathbf{B})$. This matrix is often denoted by $\mathbf{M}$. If the equation $n = r$ holds true, we call the n-port *ordinary.*

This representation allows the use of combinatorial tools in the analysis of these devices. However, it should only be used in cases where we care only about the properties of the device which are reflected by the relations of the port currents and voltages since we do not have any information about how the ports are actually connected inside the device. Also, since we considered the network equations to be homogeneous – essentially short circuiting any voltage sources, and open-circuiting any current sources inside the device –

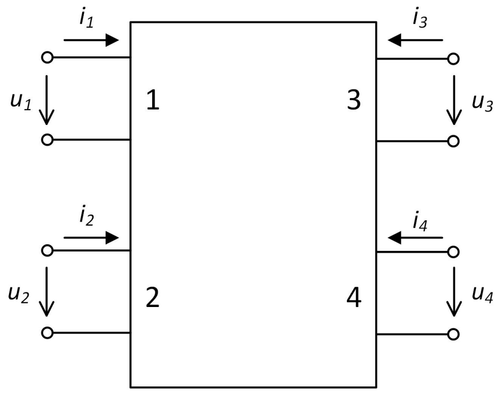


Figure 1: A 4-port

this approach is only suitable for making general statements about the solvability of the network.

2.2 The interconnection of multiports, and unique solvability

Multiports can be interconnected along a network graph, an example of this is presented in Figure 2. In this section we suppose that the multiports to be interconneted were ordinary.

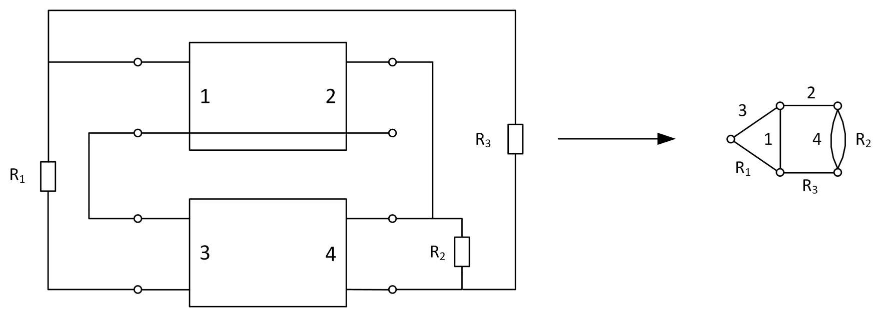


Figure 2: Two 2-ports and three resistances interconnected to form a network, and the graph of the interconnection

The resulting system of equations will contain the original describing equations of the multiports and the Kirchhoff equations. The Kirchhoff equations can be written in the form $\mathbf{Cu} = \mathbf{0}; \mathbf{Qi} = \mathbf{0}$, where $\mathbf{C}, \mathbf{Q}$ are the circuit and cut set matrices of the graph of the interconnection, respectively. Note, that when formulating the equations this way the direction of these voltages and currents would play an important role if we wanted to calculate their exact values, however, since we only examine the solvability of the network the direction of these edges is unimportant. The original describing equations of the multiports to be interconnected can be gathered in a matrix $\mathbf{A}$, where this matrix is block-diagonal and each block contains the describing equations of one of the original multiports.

If our network graph has e edges, then the solvability problem has $2e$ unknown quantities (i.e. the voltages and currents of these edges) and $2e$ equations. The original multiports were ordinary, so $\mathbf{A}$ has e rows. Since the network graph is connected $r(\mathbf{C}) = e - v + 1$ and $r(\mathbf{Q}) = v - 1$ (where v is the number of vertices in the network graph). The network equation can be written in the form:

$$\mathbf{N} \begin{bmatrix} \mathbf{u} \\ \mathbf{i} \end{bmatrix} = \mathbf{0}$$

where

$$\mathbf{N} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} \\ \multicolumn{2}{c}{\mathbf{A}} \end{bmatrix}$$

is a $2e \times 2e$ matrix. This network is uniquely solvable if and only if $r(\mathbf{N}) = 2e$. To check this condition, numerical methods are insufficient, since in the case of a very large $\mathbf{N}$, round-off errors may lead to wrong results, therefore a combinatorial approach is more adequate.

2.3 Term rank

Let us consider a matrix $\mathbf{M}$ with n columns and k rows. The term rank of $\mathbf{M}$ is the maximum number of nonzero entries, not two in the same column or row, and is denoted by $r_t(\mathbf{M})$. The determinant of a square submatrix with dimensions $n \times n$ is the signed sum of the *expansion members*, which can be written in the form $a_{1k_1}a_{2k_2}...a_{nk_n}$, where $k_1, k_2, ..., k_n$ is a permutation of the numbers $1, 2, ..., n$. If $r(\mathbf{M}) = k$, there must be a $k \times k$ submatrix $\mathbf{M}_0$ with $det(\mathbf{M_0}) \neq 0$. Since $det(\mathbf{M_0})$ is the sum of the expansion members at least one of them must be nonzero, so there have to be k nonzero entries which are all in different rows and columns. For that reason, one can observe that

$$r_t(\mathbf{M}) \geq r(\mathbf{M}) \textit{ for every matrix } \mathbf{M}.$$

Equality in general cannot hold true. Consider the matrix $\mathbf{C} = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, where a, b, c and d are nonzero entries. Clearly, $r_t(\mathbf{C}) = 2$, however, if the equation $ad = bc$ holds, $r(\mathbf{C}) = 1$, because the nonzero expansion members cancel out each other. This can only occur if there is some algebraic dependence between the entries of the matrix.

2.4 Algebraic dependence

Let us consider two fields L and K, where $K \subset L$. If $x \in L - K$, then $K(x)$ is the field K extended by x. If y_1, y_2 are further elements of $L - K$, then $y_1 \in K(x)$ and $y_2 \notin K(x)$ are both possible.

For example, let $L = \mathbb{R}$ (that is the field of real numbers) and its subfield $K = \mathbb{Q}$ (the field of rationals). Then in the case of $x = \sqrt{2}$, $\sqrt{8} \in K(x)$, but $\sqrt{3} \notin K(x)$. The first statement is trivial since $\sqrt{8} = 2\sqrt{2}$, so $\sqrt{8}$ must be contained in every field containing $\mathbb{Q}$ and $\sqrt{2}$. The smallest field containing $\mathbb{Q}$ and $\sqrt{2}$ is the field of numbers in the form $a+b\sqrt{2}$, where $a, b \in \mathbb{Q}$. To prove, that $\sqrt{3}$ is not in this field, we must show that it does not arise in this form. If we add two numbers in the form $a+b\sqrt{2}$ the result will clearly be in the same form. This is obviously also true in the case of subtraction and multiplication. Division is also possible if at least one of a and b is nonzero since $(a+b\sqrt{2})^{-1} = \frac{a-b\sqrt{2}}{a^2-2b^2}$. Let us assume that $\sqrt{3} = a+b\sqrt{2}$. Clearly, a and b must be nonzero for this to be possible. Then $3 = a^2+2b^2+2ab\sqrt{2}$, hence $\sqrt{2} = \frac{3+a^2+2b^2}{2ab}$, contradicting the irrationality of $\sqrt{2}$ (recall that $a, b \in \mathbb{Q}$). Therefore $\sqrt{3}$ can not be in the field $K(x)$.

In this case we extended $\mathbb{Q}$ with a number which is *algebraic* over it, which means there is a nonidentically zero polynomial with coefficients from $\mathbb{Q}$ that has $\sqrt{2}$ among its roots. Nonalgebraic numbers are *transcendental* over $\mathbb{Q}$. If we consider $\mathbb{Q}$ extended by a transcendental number, for example π, we obtain a very different field from $\mathbb{Q}(\sqrt{2})$. $\mathbb{Q}(\pi)$ has to contain all numbers in the form $(\sum\limits_{i=0}^{k} a_i\pi^i)/(\sum\limits_{i=0}^{l} a_j\pi^j)$, where $k, l \in \mathbb{N}$, every $a_i, b_j \in \mathbb{Q}$, and at least one of the b_j's is nonzero. This is due to π being transcendental over $\mathbb{Q}$, hence none of its powers can arise as a lower degree polynomial of π as they can in the case of $\sqrt{2}$.

We can now formulate the definition of algebraic independency. Two elements $x, y \in L$ are *algebraically independent* over K if x is transcendental over $K(y)$ and y is transcendental over $K(x)$. Generally, the elements $x_1, x_2, ... \in L$ are algebraically independent over K if any one of them is transcendental over the smallest field containing K and all the other x_i's. This also means that the elements $x_1, x_2, ...$ are algebraically independent over K if and only if they do not satisfy any nonidentically zero polynomial with coefficients from K.

Recall, that the term rank and the rank of a matrix $\mathbf{M}$ were not the same, since cancellations can occur among the entries of $\mathbf{M}$. However, if $K \subset L$, and the nonzero entries of $\mathbf{M}$ are from $L - K$ and are algebraically independent over K, then $r(\mathbf{M}) = r_t(\mathbf{M})$. [1] [3]

3 The genericity assumption

The solvability of active linear networks is one of the main problems in electrical network theory. Since the results of Kirchhoff authors gave various necessary and sufficient conditions for the unique solvability of differrent classes of devices, but most of those conditions were only necessary but not sufficient unless the generality of some parameters were assumed. The aim of this section is to highlight the importance of this genericity assumption by presenting the different results which led to its precise formulation.

In this section the active linear networks are considered as the interconnection of ordinary n-ports as described in Section 2.2. First, two models are presented for the unique solvability problem of these networks. These models were created based on the models of different authors with the goal to unify their results and highlight the not so obvious differences between them. [4]

3.1 The first model

This is the method of [5, 6, 7] slightly modified and extended by [8]. It is also equivalent to [9].

Let $G_1 = (U_1, V_1, W_1)$ be a bipartite graph, with vertex set $U_1 \cup V_1$, and edge set W_1. Let $U_1 = \{1, 2, ..., e\}$ and $V_1 = \{u_1, u_2, ...u_e, i_1, i_2, ..., i_e\} = E^u \cup E^i$, where the edge set E of the network graph G is imagined in 2 copies E^u and E^i corresponding to the currents and the voltages of the ports. Each edge of W_1 connects one vertex from U_1 with one vertex from V_1 in the following way. A vertex from V_1 is connected to $i \in U_1$ if in the matrix $\mathbf{A}$ in the column corresponding to the vertex from V_1 the i-th element is nonzero. We also define two matroids. Let the first one, $\mathcal{M}'_1$, be the free matroid over the set U_1. Furthermore

$$\mathcal{M}''_1 = (E^u, \mathcal{M}(G)) \oplus (E^i, \mathcal{M}^*(G))$$

where $\mathcal{M}(G)$ is the cycle matroid of the network graph G.

3.2 The second model

This model is very similar to the first one. It is a straightforward modification of the method presented in [10, 11] It also represents this interconnection with a bipartite graph and two matroids, however, there are some key differences.

Let $G_2 = (U_2, V_2, W_2)$ be a bipartite graph, with vertex set $U_2 \cup V_2$, and edge set W_2. Let U_2 and V_2 be two disjoint copies of $E^u \cup E^i$. Each edge of W_2 connects one vertex from U_2 with the corresponding one from V_2. $\mathcal{M}'_2$ is defined over the underlying set $S = E^u \cup E^i$ in a way that a subset $X \subseteq S$ is independent if and only if the columns of matrix $\mathbf{A}$ (defined in 2.2) corresponding to the elements of X are linearly independent. $\mathcal{M}''_2$ is exactly the same as it is in the first model, that is:

$$\mathcal{M}''_1 = (E^u, \mathcal{M}(G)) \oplus (E^i, \mathcal{M}^*(G)).$$

3.3 Unique solvability in these models

In both models, the authors gave the same condition. The network has a unique solution if and only if (in the ith model) the graph G_i and the matroids $\mathcal{M}'_i, \mathcal{M}''_i$ gave and independent matching containing e edges. A subset $X_i \subseteq W_i$ is called an independent matching if the following four conditions are all met:

1. Different edges of X_i are incident to different vertices of U_i.
2. The vertices in U_i incident to edges of X_i form an independent set in the matroid $\mathcal{M}'_i$.
3. Different edges of X_i are incident to different vertices of V_i.
4. The vertices in V_i incident to edges of X_i form an independent set in the matroid $\mathcal{M}''_i$.

The existence of such matching can be checked in polynomial time using the matroid partition algorithm [12].

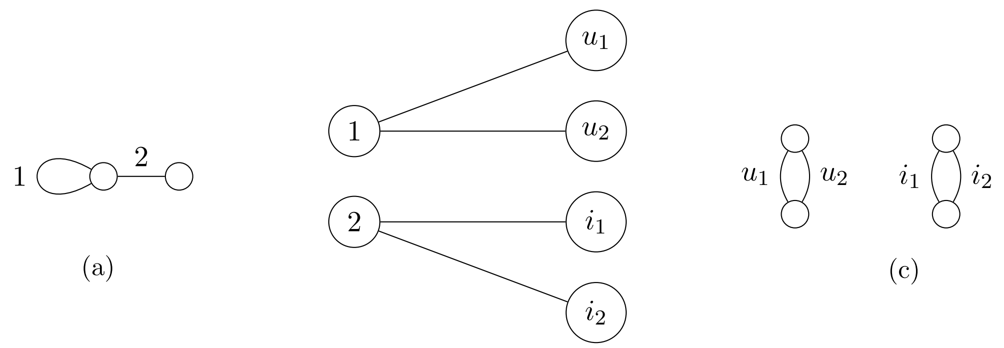
The necessity and sufficiency of the above condition will be discussed further in the following sections.

3.4 Two examples

Consider the transformer with the ratio of k terminated on one end with an open circuit and a short circuit on the other. Then the matrix $\mathbf{A}$ will look like this:

$$\mathbf{A} = \begin{bmatrix} -k & 1 & 0 & 0 \\ 0 & 0 & 1 & k \end{bmatrix}$$

and the graph of the interconnection is shown on Figure 3a. We expect to find a maximal independent matching in both models since this network is uniquely solvable. The bipartite graph of the first model is shown on 3b. In this model the edges $(1, u_2)$ and $(2, i_1)$ form an independent matching since edge 1 is cut-set free and edge 2 is circuit free in G.



(a) (b) (c)

Figure 3: First example

In the second model the matroid $\mathcal{M}_2'$ is the cycle matroid of the graph in Figure 3c. The edges of G_2 corresponding to u_2 and i_1 form and independent matching since edge 1 is cutset-free and edge 2 is circuit-free in G and they are also independent in $\mathcal{M}'_2$.

For the second example change the network graph to the one shown on Figure 4, short circuiting both terminals of the transformer and leave $\mathbf{A}$ the same. This network is obviously singular, so there should be no independent matching containing $2e$ edges in either model. In the first model the graph G_1 stays the same, but now u_1 or u_2 cannot be in an independent matching, since they are not circuit-free in G, therefore, because the bipartite graph is the same, no independent matching exists.

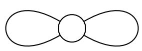


Figure 4: The network graph of the second example

In the second model u_1 and u_2 cannot be in an an independent matching due to the same reasons (recall, that $\mathcal{M}''_1$ and $\mathcal{M}''_2$ are identical), but i_1 and i_2 form a circuit in $\mathcal{M}'_2$, so there is no maximal independent matching in this model either.

3.5 The necessity of the condition

In both models the existence of a maximal independent matching containing e edges is a necessary condition of unique solvability. The proof of this statement along with the descriptions of the models can be found in [4], but is also presented here with a bit more explanation.

Let us assume that our network is uniquely solvable. This means that $det(\mathbf{N}) \neq 0$. Let us consider one of the nonzero members of the Laplace expansion of this determinant. Recall, that $\mathbf{N}$ was a $2e \times 2e$ matrix with the first e columns corresponding to the voltages and the second e columns corresponding to the currents of the ports. Then, this nonzero expansion member arises as the product of the determinants of the shaded submatrices of

$\mathbf{N}$ as shown in Figure 5. Let us denote these submatrices with C', Q' and A', respectively. Since $det(\mathbf{N}) \neq 0$, Q' has to represent a maximal circuit-free subgraph (i.e. a tree since G is connected) of the network graph G and C' must represent the remaining edges (i.e. a cotree of G). Also, the columns of A' must be linearly independent. Hence, by definition, the voltages and currents corresponding to the columns of A' are independent in the matroid $\mathcal{M}_2'$. Also, the columns of A' which correspond to voltages represent a tree (since C' represented a cotree) and the columns corresponding to currents represent a cotree (since Q' represented a tree) in the network graph G. Therefore these columns are also independent in $\mathcal{M}_2''$. Thus, the columns of A' correspond to an independent matching containing e edges in the second model proving that this is a necessary condition of unique solvability.

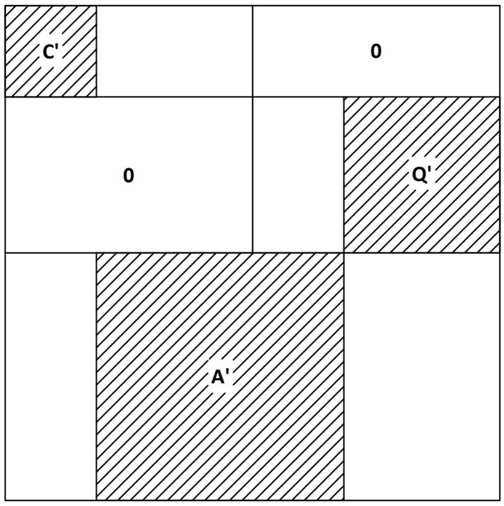


Figure 5: An expansion member $det(\mathbf{N})$ after rearrangement of the columns

Now expand the determinant of A' to be the sum of the products of nonzero entries. Pick one of the nonzero members of this expansion. This expansion member corresponds to e different entries in A', not two in the same column or row, and those edges correspond to edges of the bipartite graph G_1. These vertices in U_1 incident to these edges are obviously independent in $\mathcal{M}_1'$ since it was the free matroid over U_1. The vertices of V_1 incident to these edges are the ones which the columns of the matrix A' correspond to. These are independent in $\mathcal{M}_1''$ due to the exact same reason they were independent in $\mathcal{M}_2''$, so these edges of the graph G_1 form an independent matching containing e edges in the first model. Therefore the necessity of the above condition is also proven in this case.

3.6 The insufficiency of the condition

In the previous section we saw that the existence of an independent matching containing e edges means in both models that there is a nonzero member in the expansion of $det(\mathbf{N})$, that is $r_t(\mathbf{N}) = 2e$. However, as discussed in Section 2.3, this does not necessarily mean that $r(\mathbf{N}) = 2e$, since cancellations can occur. Hence the existence of an independent matching containing e edges is not a sufficient condition of unique solvability as shown by the following example.

Consider the network of Figure 6. It is obviously singular since

$$\mathbf{N} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & R & 0 \\ 1 & 0 & 0 & -R \end{bmatrix}.$$

However, the condition is satisfied in both models. In the first model, the edges $(1, u_1)$ and $(2, i_1)$ of graph G_1 form an independent matching. In the second model, the edges corresponding to u_1 and i_1 also form a maximal independent matching.

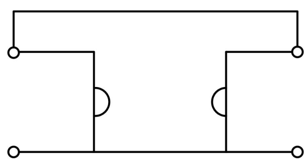


Figure 6: A gyrator with its ports in parallel

In this example, the parameters of the device were canceled out by the entries of the matrices $\mathbf{C}$ and $\mathbf{Q}$. To make the above condition sufficient, we need to somehow prevent these cancellations from occurring. Let us assume that all of the paramaters of the devices to be interconnected are algebraically independent transcendentals over the field which the entries of $\mathbf{C}$ and $\mathbf{Q}$ are selected from. If we add this assumption, the parameters cannot cancel out each other. This assumption is called the *genericity assumption* in electrical network theory. Now with this addition, we can formulate a necessary and sufficient condition for the unique solvability of a network.

If the genericity assumption holds, the existence of a maximal independent matching containing e edges is also sufficient for the unique solvability of the network in both models.

3.7 Different types of genericity

So far the first and the second model behaved identically. Consider the following example, also found in [4]. A network is given by the following description:

$$\mathbf{N} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 2 \\ 0 & -1 & 2 & 4 \end{bmatrix}.$$

It is really easy to see that $r(\mathbf{N}) = 3$, therefore this network is singular. However, in the first model the edges $(1, i_1)$ and $(2, i_2)$ form a maximal independent matching since 1 and 2 are independent in the free matroid $\mathcal{M}_1'$ and the network graph consists of two loops as indicated by the first two rows of $\mathbf{N}$, so i_1 and i_2 are independent in $\mathcal{M}_1''$. Now, rearrange the equations in the following form:

$$\mathbf{N} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & -1 & 2 & 4 \end{bmatrix}.$$

This matrix clearly represents the same network (recall that a multiport can have several different matrix descriptions) since the equations were obtained by multiplying the third row by 2 and subtracting the fourth row from the third. Now the graph G_1 is shown on Figure 7.

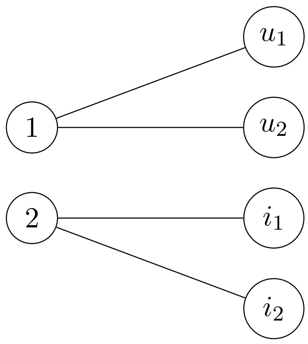


Figure 7: The graph G_1 of the example

The network graph is the same – two loops – which means the edges of an independent matching cannot be incident to u_1 or u_2, therefore no independent matching exists. Note, that in the second model there was no independent matching in either cases. Actually, the second model is the same regardless of the actual matrix description, it is only influenced by the linear dependencies of the columns which must be the same in all the matrices describing a particular network.

In the previous section we saw that, if we consider the describing parameters of the multiports to be interconnected as algebraically independent transcendentals over the field which the entries of $\mathbf{C}$ and $\mathbf{Q}$ are selected from (which means that all algebraic relations are forbidden both within the matrices of single multiports and among the matrices of different multiports), then the existence of a maximal independent matching is a necessary and sufficient condition for the unique solvability of the network.

This assumption is often referred to as the *strong genericity assumption* [6], and it is fairly realistic, because these entries represent physical parameters and due to technological constraints a predetermined algebraic relation between them is highly unlikely. However, there are multiple problems with it. The first one is that it is too restrictive. It forbids the inclusion of important devices such as ideal transformers or gyrators since their parameters (transfer ratio, gyration resistance) arise in two equations each. The second problem is that it refers to the actual matrix description of the multiports. Since it is possible to construct different matrix descriptions of the same devices so that this assumption holds in one, but not in the other, we may exclude solvable networks if they are given with particular matrices. Note, that since the first model also depends on the actual matrix description as shown in the previous example, it needs this assumption for the above condition to be sufficient for unique solvability.

These difficulties can be avoided by using the *weak genericity assumption* [10], which states that among the nonzero entries of the multiport matrices the only possible algebraic relations are the ones reflected by the structure of the matroid $\mathcal{M}$ modeling the multiports ($\mathcal{M}(S, F)$ is the matroid modeling a multiport, S is the set of the columns of the matrix $\mathbf{M}$ and $X \subseteq S$ is in F if the columns of are linearly independent in $\mathbf{M}$). This means that certain relations within the matrices of a single multiport are allowed while other algebraic relations between the matrices of different multiports are still forbidden. This assumption does not depend on the actual descriptions of those multiports, as all of the describing matrices of a particular multiport have the same linear dependencies among their columns, hence the assumption allows exactly the same algebraic relations in every case. If we add this assumption to our condition and use the second model the condition becomes sufficient.

3.8 Genericity in statics

In statics, a 2- or 3-dimensional bar and joint framework F can be described with a graph $G(F)$. The points of G correspond to the joints of F and two points are adjacent in G if there is a rod in F between the corresponding joints. Sometimes the rigidity of this network can be determined from this graph alone, but other times it depends on the length of the rods as well. For that reason, a more precise description is needed. A framework can be described with the equation $\mathbf{Au} = \mathbf{0}$. $\mathbf{A}$ is a $e \times kv$ matrix, where e is the number of edges in G (that is the number of bars in F), k is the dimension of the space in which F is considered and v is the number of vertices in G (that is the number of joints in F). This matrix contains the relative position of each joint, therefore determining the length of the rods, and the vector $\mathbf{u}$ contains the velocities of each joint. For further information on this description, the reader is referred to [1], in this present paper it is enough to understand that $\mathbf{A}$ holds all information needed to determine the rigidity of the framework it describes.

A framework is called rigid if $r(\mathbf{A})$ is $2v - 3$ in the 2-dimensional and $3v - 6$ in the 3-dimensional case. This may be intuitive since each joint has 2 or 3 degrees of freedom respectively and the whole *rigid* framework has 3 degrees of freedom in the 2-, and 6 in the 3-dimensional space.

Maxwell already knew in 1864 that a 2-dimensional frame requires at least $2v - 3$ bars to be rigid [13]. However, to make this condition sufficient a slight modification and the addition of the genericity assumption (all the coordinates of the joints are algebraically independent transcendentals over $\mathbb{Q}$) was needed. The applicability of such assumption is further discussed in [1].

If a planar framework F has v joints and $e = 2v - 3$ rods, a necessary and sufficient condition for its rigidity is that $e' \leq 2v' - 3$ holds for every subsystem of F having v' joints and e' rods [14]. This condition is only sufficient, if the genericity of the coordinates of the joints is assumed. This condition cannot be checked in polynomial time. Lovász and

Yemini presented another condition which can be checked in polynomial time, however it still requires the coordinates of the joints to be general [15]. This genericity is needed, because a predetermined algebraic relation between the coordinates can reduce the rank of $\mathbf{A}$. For example, a framework isomorphic to the Kuratowski graph $K_{3,3}$ is infinitesimally rigid, unless the six joints are on a common conic section – this condition means that there is a quadratic equation among the 12 coordinates of the six points [16].

4 Further properties of the genericity assumption

In this section some more recent results are presented on how the genericity assumption influences the rank of a matrix [17].

As seen in Section 3, the genericity assumption is often applied to prevent the decrease of the rank of a matrix due to algebraic relations between parameters. This was necessary to study the unique solvability of networks (or rigidity of frameworks) without the need to consider the actual parameters. Knowing this, it is intuitive to assume that dropping the genericity assumption can only decrease the rank of a matrix since it allows expansion members from cancelling out each other. However, if we interconnect multiports to form another multiport we may obtain a surprising result: the rank of this new multiport can also increase if we drop genericity.

To make this discussion more convenient, the interconnection of multiports will be described by matroids instead of matrices. The matroid describing the multiport denoted $\mathcal{M}(\mathbf{M})$ is the column space matroid of $\mathbf{M}$ as described in Section 3.7. Observe, that this matroid is the same, regardless of the actual matrix description of the multiport, however it no longer contains any quantitative information about the currents and voltages of the ports.

Multiports can be interconnected along a network graph $G(E, V)$ to form another multiport, an example of this is presented in Figure 8. Notice that in this case, as opposed to the network of Figure 2, we designated some of the points of the resulting network to be ports of a newly formed multiport.

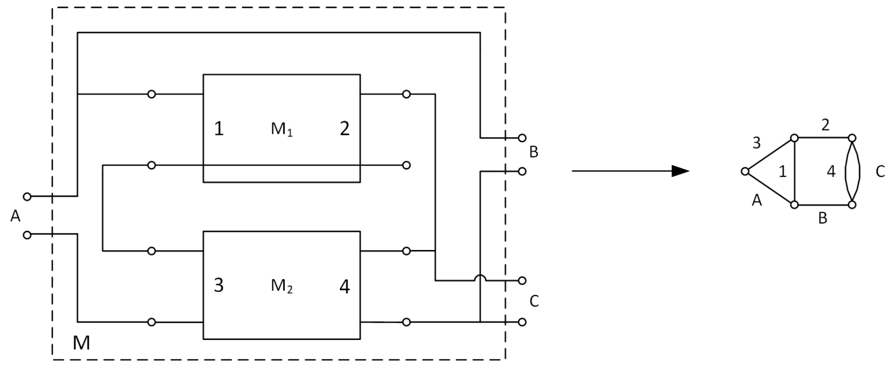


Figure 8: Two 2-ports interconnected to form a 3-port, and the graph of the interconnection

The edge set of the interconnection graph is the union of the set E_{Int} of the internal edges (the edges corresponding to the ports of the original multiport) and the set E_{Ext} of the external edges (the edges corresponding to the ports of the resulting multiport). Since each port has a voltage and a current, let E^u and E^i denote the set of all the voltages and that of all the currents, respectively, similar to Sections 3.1 and 3.2. These sets can be further decomposed as follows: $E^u = E^u_{Int} \cup E^u_{Ext}$ and $E^i = E^i_{Int} \cup E^i_{Ext}$.

For the sake of easier notation, the matroids $\mathcal{M}''_1$ and $\mathcal{M}''_2$ of Sections 3.1 and 3.2 are denoted by $\mathcal{G}$ in this section. That is

$$\mathcal{G} = (E^u, \mathcal{M}(G)) \oplus (E^i, \mathcal{M}^*(G))$$

Let $\mathcal{A}'$ denote the direct sum of the matroids of the multiports to be interconnected on the set E_{Int}. Extend $\mathcal{A}'$ with loops on the set E_{Ext} to obtain a matroid $\mathcal{A}$. This way both matroids are defined over the sets $E^u \cup E^i$.

If the genericity assumption holds, the matroid of the new multiport is

$$\mathcal{M}(\mathbf{M}) = (\mathcal{G} \vee \mathcal{A})/(E^u_{Int} \cup E^i_{Int}) \text{ [1].}$$

If we drop genericity an occurring cancellation usually decreases the rank of the multiport. However, if this cancellation happens to be in the set $E^u_{Int} \cup E^i_{Int}$, then – since the rank of the set to be contracted decreases – the rank of the final multiport can also increase. This can be illustrated using the network of Carlin an Youla on Figure 9 [18] [19].

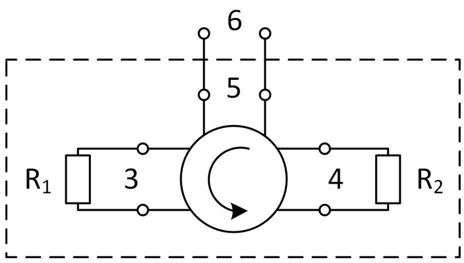


Figure 9: The circulator network of Carlin and Youla

In the generic case, there are 11 linearly independent equations describing the 10 parameters of the original multiports and the 2 parameters of the new one. The matroid of the 3-port circulator is isomorphic to the cycle matroid of K_4 (that is the complete graph with 4 vertices), see [4]. The matroids $\mathcal{G}$ and $\mathcal{A}$ are the cycle matroids of Figure 10a and 10b, respectively. The union of these is the uniform matroid $\mathcal{U}_{11,12}$, in which all proper subsets are independent and its graph is the circuit of length 12. Hence, after the contraction of edges with indexes 1–5 we obtain a length 2 circuit of u_6 and i_6, which corresponds to a resistor as expected in the generic case.

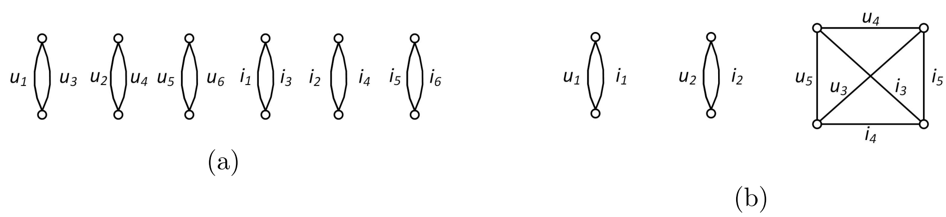


Figure 10: The graphs of $\mathcal{G}$ and $\mathcal{A}$ of the example

Now drop genericity and put $R_1 = 1$ and $R_2 = -1$. Now the number of linearly independent equations are 10. The column space matroid of the coefficient matrix is different from $\mathcal{G} \vee \mathcal{A}$ this time. Recall, that neither $\mathcal{G}$ nor $\mathcal{A}$ takes into account the actual entries of the coefficient matrix since $\mathcal{G}$ is obtained from the network graph and $\mathcal{A}$ just gathers the structure of the individual multiports, so they do not change if we add dependencies between the parameters of different original multiports. This new matroid is graphic again, see Figure 11 for its representation. If we contract the edges with indices 1–5, we obtain two loops, corresponding to the norator – a *decrease* of the final rank, as expected.

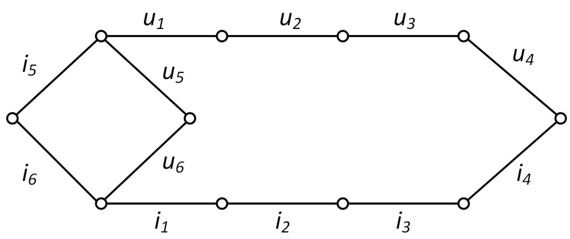


Figure 11: The graph of the matroid of the network in the case $R_1 = 1, R_2 = -1$

If we put $R_1 = -1$ and $R_2 = 1$, all 11 equations remain linearly independent, however the matroid of the coefficient matrix changes again. It becomes the direct sum of a length 8 circuit and 4 bridges corresponding to u_5, u_6, i_5 and i_6, so the subgraph of the edges corresponding to the internal parameters is not circuit-free and after contracting them, we obtain two bridges, corresponding to the nullator. Due to the subtracted subgraph containing a circuit, thus having decreased rank, the rank of the resulting 1-port *increased.*

Consider some other special cases. If $R_1 + R_2 = 0$ but $R_1R_2 \neq -1$, then the rank of the coefficient matrix is still 11, however, the matroid becomes the direct sum of a length 10 circuit and two bridges corresponding to i_5 and i_6. After contraction we obtain a loop representing u_6 and a bridge representing i_6, which correspond to the matroid of an open circuit.

If $R_1 + R_2 \neq 0$ but $R_1R_2 = -1$, the number of linearly independent equations is 11, and the matroid is the same as it was in the previous example, the only difference being that the bridges now correspond to u_5 and u_6. After contraction we obtain a bridge representing u_6 and a loop representing i_6, corresponding to the matroid of a closed circuit.

If none of the above relations hold, the matroid can still differ from $\mathcal{U}_{11,12}$. For example, if $R_1 = 1$ and $R_2 = 1$, the matroid is a circuit of 8 with bridges corresponding to u_2, u_4, i_2 and i_4, however, after contraction we obtain the result of the generic case.

For the complete understanding of these remarks, from an engineering standpoint it might be instructive to solve the system of 11 equations without using matroid theory to obtain the relation between the voltage and current of port 6 (i.e. the resistance of the port) as a function of the values of the resistances terminating port 3 and 4. This way we obtain the form:

$$u_6 = \frac{R_1 R_2 + 1}{R_1 + R_2} i_6$$

We can plot this resistance value as a surface to visualize the network, see Figure 12. Aside from the generic case, this further illustrates the above results since it is immediately clear that if $R_1 + R_2 = 0$, the resistance of this port is infinite, therefore it behaves like an open circuit, and if $R_1 R_2 = -1$, the resistance of this port is zero, therefore it behaves like a closed circuit.

This is in correspondence with the remarks of [19], that nullators and norators are singular elements, and if some circuit element in the equivalent structure for a nullator or norator is changed slightly, the terminal performance will no longer be similar to that of the nullator or norator. It also illustrates that they are very similar to each other and arise in cases where a network simultaneously behaves like a short circuit and an open circuit.

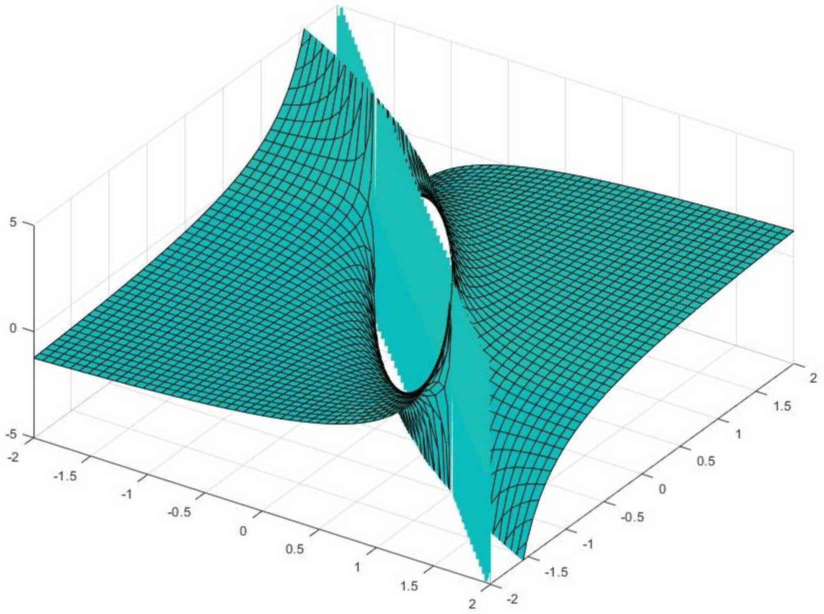


Figure 12: The resistance of port 6 as a function of R_1 and R_2

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