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Pricing auto-workout mortgage products by associated default risks

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Abstract

The two main causes of mortgage payment difficulties are rising house prices and a temporary loss of household income. Currently available bank products handle these problems only when the mortgage is already delinquent (usually for a long time). The so-called workout process that starts at this point usually results in either loan restructuring or termination. A number of ideas and initiatives are present in the literature which brings this expensive workout process earlier and in an automatic fashion, thereby making it more predictable and less expensive for both sides. By using an embedded optionality in the mortgage product it is possible to reduce scheduled payments or the balance itself with the decline of the value of the property offered as collateral. The paper aims to create a theoretical model and its simulations on mortgage products with an above mentioned embedded optionality and to propose a fair pricing method.

1 Introduction

1.1 Mortgages

Mortgages are debt instruments secured by a collateral which is in most cases a real estate property. Mortgages are used by both individuals and businesses to purchase real estate without having the full capital needed for the transaction. A mortgage is a contract between the lender and the borrower in which they state that the borrower is obliged to repay the borrowed amount named principal plus interest.

Usual mortgages have a 10-30 years scheduled life span named maturity and payments are usually scheduled monthly. If the borrower fails to make the payments the bank has a claim on the property therefore payment delinquencies usually result in a long and costly foreclosure procedure. This procedure is not favored by either of the parties because it tends to result in equity loss for both the lender and the borrower. Over the last few years much attention has been drawn to the complex nature of mortgage defaults and its drivers.

1.2 The U.S. housing boom and the following crisis

In the early years of the past decade a rapid expansion took place in the U.S. housing market. Housing prices were on an evident upwards trend and the general public opinion was that it will continue going up in the future. The ever accelerating housing price increase was examined by many papers and books. Robert Shiller first released his book[1] on overvalued stocks while in the middle of the dot-com bubble.

Later in 2004 Shiller worked on papers[2] examining the housing market at the time drawing attention to the fundamentals that may indicate a developing bubble. In the 2nd edition of his book[3] he wrote that the real estate bubble may soon burst. Figure (1) from the 2nd edition of the book shows how the growth in population does not indicate the rapid change in the supply and demand equilibrium which took place.

A strong indication of a fundamental driver behind housing appreciation is the evolution of mortgage interest rates which were declining for more than 15 years as seen in Figure (2).

It has been clear that two of many contributors of the boom were ever lower interest rates and and ever lower mortgage standards which made financing easier. It is important to note that although easier financing makes housing more acces-

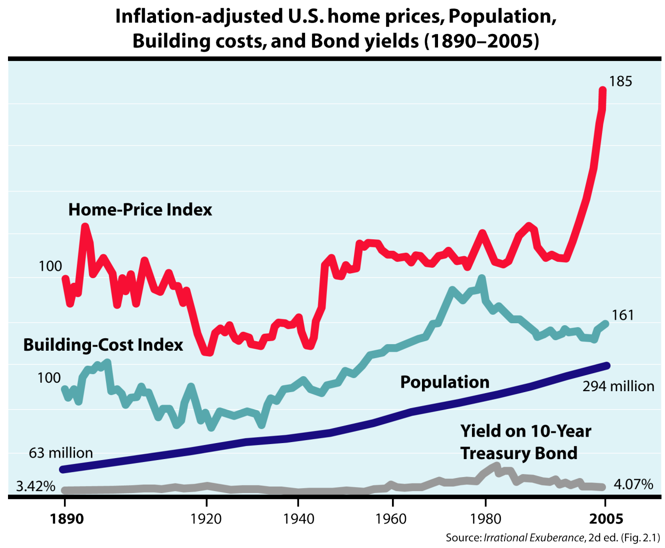

Figure 1: The rapid housing price appreciation was not due to a population increase[3]

sible it does not make it more affordable since easy financing raises the demand for housing creating a new equilibrium with higher house prices and thus making it more and more expensive.

Housing prices peaked in early 2006 and started a rapid decline in between 2006 and 2008. In 2008 the Case-Shiller composite home price index reported its largest drop in its history. Mortgage delinquencies were high in the previous years and continued to be ever higher in the coming years resulting in the credit crisis of the late 2000's. This was followed by the huge institutions reporting serious losses and some of them going bankrupt even though considered to be “too big to fail”. The resulting credit crunch was the cause of a lack of liquidity which made financing hard for businesses and resulted in the 2007-2009 recession in the U.S. The causes and the results were examined in more detail by many papers in the literature[4, 5].

1.3 Mortgage defaults

The problem of mortgage defaults was not new before the credit crisis and both risk management methods and regulations were trying to make sure that banks would not take bigger risks than they can handle. International standards like the

Figure 2: Declining mortgage rates from the early 80's

Basel II[11], the second of the Basel Accords were formulating recommendations on risk management and its implementation and were accepted and used by most banks and were also enforced by many European authorities.

These standards have in their main scope some sensitive methods for calculating credit risk capital requirements, default probabilities and how to estimate them, but they contain little information on the the interaction between macroeconomic processes and mortgage defaults.

Many new models have emerged in trying to describe these interactions with more information than just historical default rates of clusters of borrowers. Defaults also resulted in many suggestions on how loans should be modified to mitigate risk exposure. Some of these are called Auto-Workout Mortgages which I will present in detail in the following chapter after first presenting traditional mortgage constructions.

2 Traditional mortgages

The standard for most mortgages is that the borrower gets the loan amount after origination and agrees with the lender in a starting date for scheduled payments of which the most common are monthly versions. The time between origination and the first scheduled payment varies from a months to even years, however most of them have their first scheduled payments within 5 months after origination.

The maturity of the mortgage does not necessarily mean the time when there is only one final small payment left. Some mortgages have most of its repayable amount scheduled at maturity, called a balloon payment which usually gets refinanced to continue with a new contract and thus new maturity date. This difference between mortgages is called difference in amortization which can vary on a wide spectrum from interest only mortgages, where monthly payments cover only interests and have the full principal amount scheduled at maturity to fully amortizing mortgages which distribute payments evenly to cover both interest payments and a part of the principal. Most mortgages with a big payment scheduled at maturity are more popular as commercial mortgages originated for businesses who choose to own their own property instead of renting but need most of their capital for running projects.

The most popular residential mortgage constructions are fully amortizing mortgages. Most of these have a fix scheduled payment every month which does not follow inflation. The advantage of this is their simplicity and predictability for the public.

The two main types of residential mortgages differ in how their interest rate is given, namely fixed-rate mortgages (FRM), where the borrower's payments do not change and adjustable rate mortgages (ARM) where the interest rate is fixed to some market variable and varies across the lifespan of the loan. Most long term, 15- or 30-year mortgages in the U.S. were originated with a fixed rate which was popular among the public because of not having to deal with interest rate risk no matter how market interest rates changed in the future. Most adjustable rate mortgages have a so called teaser period when the interest rate is initially fixed at a usually low level below market rates but then it fluctuates with market interest rates. In many cases this results in loans that are both unpredictable and less affordable than they seem.

The most common U.S. mortgage construction is a 30 year FRM but usually this does not mean that borrowers are sure to stay for the full 30 years of the lifespan. Aside from defaulting, many borrowers choose to repay the owed amount to the bank prior to maturity, called prepayment. In these cases the borrower pays the full amount of its remaining balance, which is the remaining principal amount, without further interest payments. In fact most prepayments are made not because the borrower accumulated enough wealth to easily repay the mortgage ahead of time but because they have found another mortgage with better terms usually at a lower interest rate. Interest rates getting ever lower as seen in figure (2) made prepayments very common as it was easy to find a new mortgage on the market with lower rates. As it's not the paper's main focus, I will avoid handling prepayments as possible, with the exception of validating models on real data.

2.1 Fully amortizing Fixed Rate Mortgages

In this paper I will compare more complex mortgage products to FRMs both because they were most common in the U.S. and because they are the basic reference point due to their simplicity. Here I present some basic equations for handling FRMs. The first two concepts, the maturity T of the mortgages in years and yearly interest rate r have already been mentioned before. Payments are scheduled using compound interest, where the compounding time τ is the inverse of the number of payments per year m. The loan will stretch out on the $t = 0, \tau, 2\tau, ..., T-\tau, T$ time horizon. I will use the case when $\tau = \frac{1}{m} = \frac{1}{12}$. The initial balance B_0 will be the loan amount which the borrower receives at origination. After one time period the borrower is obliged to pay interest accountable for that period, making the total owed amount before payment $(1+r\tau)B_0$. This gives us the formula for calculating interest payments:

$$I_{t+\tau} = r\tau B_t \tag{1}$$

At each payment date the amount scheduled Q will be composed of this interest payment part I_t and a principal payment part P_t in the following way: $Q = I_t + P_t$. This brings us to a recursive formula for calculating the remaining balance after a payment as follows:

$$B_{t+\tau} = (1+r\tau)B_t - Q = B_t - P_{t+\tau} \tag{2}$$

In the case of an FRM the value of each individualP_t is set so that the mortgage will be fully amortized at maturity and the sum of all scheduled payments Q discounted to the start date $t = 0$ with the interest rate r should give back the original mortgage balance. This is the same concept as the Internal Rate of Return used to compare different investments:

$$NPV = -B_0 + \sum_{i=1}^{T/\tau} \frac{Q}{(1+r\tau)^i} = 0 \tag{3}$$

In this case it is used to calculate the value of Q and thus the values of P_t given a desired interest rate but can also be used backwards to obtain the internal rate of return of a cash flow given each of the individual payments. The internal rate of return of a mortgage fully paid back and always on time will be the original interest rate r. I will return to this formula and the concept of the Internal Rate of Return at the final step of the mortgage pricing process. With the above mentioned IRR condition we obtain the formula for the scheduled fix payments as

$$Q = B_0 \frac{r\tau}{1-(1+r\tau)^{-T/\tau}} \tag{4}$$

Using the above equations we can derive a closed formula for the evolution of the outstanding balance at time t (after payment):

$$B_t = B_0 \frac{1-(1+r\tau)^{-(T-t)/\tau}}{1-(1+r\tau)^{-T/\tau}} \tag{5}$$

We can also obtain a recursive formula for Q in the form

$$Q_{t+\tau} = B_t \frac{r\tau}{1-(1+r\tau)^{-(T-t)/\tau}} \tag{6}$$

This is mainly used to construct adjustable rate mortgages where the interest rate r will vary monthly but it will also give a good basis for describing other types of more complex mortgage constructions with adjustable payments.

Figure (3) illustrates the evolution of the balance over a 10 step period. We can see that in the initial periods most of the payments make up only for paying the interest and not much is left for paying off the principal amount and decreasing the balance itself. At later periods as the balance decreases the interest payments make up for less and less of the scheduled payments and thus most of the principal payments happen later in time.

These equations will be the basis for all mortgage constructions later in the paper modifying some of the terms present to vary periodic payments.

2.2 Mortgage related quantities

In the process of managing a bank's risk exposure there have been industry wide standards for quantitative measures of loan quality. Most of these indicators as-

FRM				Domestic currency			Falling house price index		
Period	Property price	LTV	Balance before	Principal payment	Interest payment	FRM Total payment	Balance after	Balance after eq (1.4)	Total Payment eq (1.5)
0	1000000	0	0	0	0	0	1000000	1000000	0
1	1000000	0.069029	1000000	69029	80000	149029	930971	930971	149029
2	1000000	0.074552	930971	74552	74478	149029	856419	856419	149029
3	800000	0.100645	856419	80516	68513	149029	775903	775903	149029
4	800000	0.108697	775903	86957	62072	149029	688945	688945	149029
5	500000	0.187828	688945	93914	55116	149029	595032	595032	149029
6	500000	0.202854	595032	101427	47603	149029	493605	493605	149029
7	500000	0.219082	493605	109541	39488	149029	384063	384063	149029
8	800000	0.147881	384063	118304	30725	149029	265759	265759	149029
9	800000	0.159711	265759	127769	21261	149029	137990	137990	149029
10	1100000	0.125446	137990	137990	11039	149029	0	0	149029
Total				**1000000**	**490295**	**1490295**			
NPV					**360838**	**1000000**			

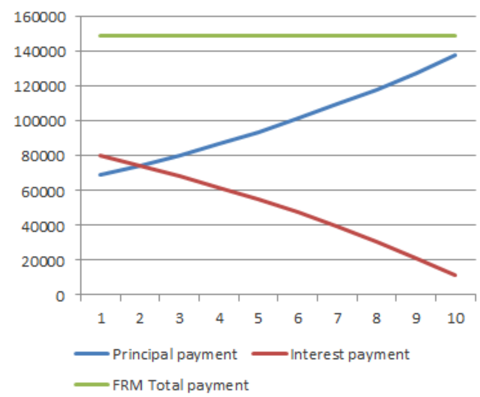

Figure 3: Evolution of an FRM up to maturity

sess how the loan amount measures up to some other variables usually known for most (potential) customers. These numbers help lenders in the initial phases of the mortgage origination to decide whether or not they should lend to a certain customer and what amount would be appropriate given his/her income and provided collateral. These indicators are also useful to monitor the loan throughout their lifespan as the customer's income will almost surely change during the long lifespan of a 10-30 year mortgage as well as property appreciation and depreciation can happen.

Loan To Value ratio and Equity

The Loan-To-Value ratio (LTV) is one of the above mentioned measures and represents the ratio of the borrower's current balance to the collateral's value, in this case the real estate property's value. At a time t I define the value of the LTV to be

$$LTV_t = \frac{B_{t-\tau}}{C_t} \tag{7}$$

where $B_{t-\tau}$ is the value of the balance in the last time period after payment and C_t is the most up to date value of the collateral or the real estate property in our case.

In general, lenders prefer lower LTVs as a high LTV would indicate that the contract with the borrower is riskier as he is less likely to absorb the losses of a default. I try to avoid the confusion of referencing multiple values at a given time t by using the values of the balance B_t to always represent the value after the scheduled payment is done. Using this definition instead of the simpler looking $\frac{B_t}{C_t}$ I can include LTV values with the most up to date property price information before the scheduled payment at time t which will be important when using it for modeling LTV dependent default decisions.

Another way of formulating the same idea is the amount of equity E_t held by the borrower in the real estate which in this case can be expressed as $E_t = C_t - B_{t-\tau}$. We can immediately see that an LTV value less than 1 results in a negative equity which means that the borrower owes more money than he is covering with the collateral and it is assumed that he is more likely to default in such a situation. Due to the rapid appreciation of real estate prices in the early 2000's banks were not very concerned about keeping this value low as both housing price trends and the continuous payments only decreased the starting LTV value with time.

Along the simple LTV it is preferred to use the Combined Loan To Value Ratio (CLTV) which is different from the simple LTV if more than one loan is collateralized by the same property. As presented in [7] there was a widespread requirement to have a 20% down payment and to have the initial $LTV_0 = 0.8$. Many borrowers lacking the amount for the down-payment could mainly chose from 2 options. They could buy Private Mortgage Insurance which would protect the bank in the case of a default but obliged the borrower to pay additional monthly fees for the insurance itself. Many borrowers however used a second mortgage on the same property dividing the loan amounts in a 80-20 fashion or in a 80-10 fashion plus a 10% down-payment. These loans were the so-called "piggyback" loans. This phenomena leads to the simple LTV calculated by the first mortgage originator being inaccurate in the sense that it doesn't represent correctly what it was meant to. In [7] we can see that insured loans were found to have lower delinquency rates than loans with additional second loans on the same property. To address this issue we can define the Cumulative Loan to Value Ratio as

$$CLTV_t = \frac{B^{(1)}_{t-\tau} + B^{(2)}_{t-\tau}}{C_t}$$

where $B^{(1)}$ and $B^{(2)}$ represent the balance of the individual loans. Although sometimes the CLTV value is known we won't have the information on scheduled payments nor amortization of the second loan. Along the paper I used CLTV values and treated possible separate loans as if they were one big loan as it represents a better picture on the borrower's indebtedness as just using the simple LTV.

Estimating LTV after origination

At the time of loan origination the property given as collateral undergoes a valuation process where the fair market value is given for that time. After origination however, we do not directly observe the most up to date value of the property and we estimate its value by using the property value at $t = 0$ and the most current value of a House Price Index. The estimated value of the property at time t is given as follows:

$$\hat{C}_t = \frac{H_t}{H_0} C_0 \tag{8}$$

where H_t and H_0 are the House Price Index values at times t and 0 respectively.

Foreclosure costs

After a loan is several months delinquent it is said to be defaulted and usually ends up in a costly foreclosure process including legal fees, maintaining hazard insurance and usually property taxes and home maintenance fees until the house is sold. This usually results in several fees accumulating which gets billed to the borrower and in most cases drawn from price of the real estate after sale. Most bank sold homes are also sold below market price because the bank doesn't want to hold on to real estate properties for a very long time until someone offers a decent price but wants the capital back for further investments. Estimated foreclosure costs given in the literature[6] can be up to 40% of the pre-foreclosure value of the property. According to Standard & Poor's 2008 reports the costs of a typical foreclosure are around 26% of the loan amount. I will use this value throughout simulations. These values also warn us about the danger of high LTV values even below the 1 threshold.

3 Auto Workout Mortgages

Along examining the triggers of mortgage defaults there has been much effort done in rethinking of the conditions and the structure of mortgage products themselves. A key finding in the literature that while an FRM has the ability to take the interest rate risk from the borrower and put it to the side of the lender, the application of a similar strategy to the housing price risk was never considered before the crisis.

A sharp downturn in housing prices can result masses of borrowers having to face negative equity and thus challenged by the fact that not considering ruined credit scores and other side effects it would be a financially rational decision to default. There is evidence[5, 10] that a high LTV or CLTV ratio is highly correlated with the frequency of defaults although the side effects result that this rationality can not be applied as directly as modeling the borrower's behavior as making a decision on when is it optimal to default and thus to maximize the resulting gains. It is clear however that for this rational decision to be taken into consideration we need a necessary condition of a negative equity.

In essence, in traditional mortgage constructions all of the house price risk is beared by the borrower. Financial institutions however would be more economically efficient bearers of this risks and thus some or all of it should be transferred to them. In the following subsections I will present two proposed mortgage constructions from the family of Auto Workout Mortgages (AWMs) and compare them by demonstrating their behavior under a simple scenario and I will also examine a few of my ideas for modifying them in ways that would be interesting.

The main idea behind all AWMs is to embed some kind optionality in the mortgage contract that would result in a positive payoff in certain cases and results a lower balance or lower scheduled payments.

3.1 Adjustable Balance Mortgages

The mortgage constructions proposed by Ambrose and Buttimer [9] called Adjustable Balance Mortgages (ABMs) which directly address the problem of a borrower faced with negative equity in times of house price depreciation. It's proposal is that instead of relying on external frictions such as a ruined credit score and the inconveniences of the foreclosure process to motivate borrowers not to default, the mortgage contract itself should automatically reset the principal balance to the minimum of the original scheduled balance or the (estimated) value of the house.

The structure of the mortgage itself is based completely of the FRM, also having a fixed interest rate and is also fully amortizing. Given that no loans are originated with an LTV value higher than 1 the ABM behaves exactly like the FRM both at origination and up until the first time of experiencing negative equity. The only difference between the FRM and the ABM is that the balance is capped to the value of the house and the following scheduled payments are calculated using the new lowered balance. After property prices climb bank the balance is reset to the original scheduled values of that of an FRM. All payment reductions during a time when the balance cap is active form the loss of the lender.

The estimation of the house price during the life time of the mortgage is done as previously presented in Equation (8). I will present the equations describing the ABM as stated in[16]. The above mentioned balance cap is given simply by referring to the balance of the corresponding FRM:

$$B_t^{ABM} = min(B_t^{FRM}, \hat{C}_t) \tag{9}$$

The scheduled payments of an ABM is calculated with modifying the recursive formula presented for the FRM at Equation (6) and is given as

$$Q_{t+\tau}^{ABM} = min(B_t^{FRM}, \hat{C}_{t+\tau}) \frac{r\tau}{1-(1+r\tau)^{-(T-t)/\tau}} \tag{10}$$

where in both cases B_t^{FRM} is the original scheduled balance of the corresponding FRM seen at Equation(5). It is interesting to note that if the price of the house declines more than the scheduled principal payments at the adjustment period then the cap is fixing the balance to the same house price even after the next payment of the borrower. This means that the balance is not reduced more than to the value of the house at this phase while the cap is active even though adjusted payments are still being made. The mortgage returns to the original FRM schedule after the originally scheduled balance is less than the house price. The definition of the ABM has references to the related FRM balance B^{FRM} even after payment adjustments are made and does not keep track of real interest and principal payments. One consequence of this is that if balance reduction was active during the lifetime of the mortgage then we don't just lose interest through the payment reductions but we also lose some of the principal.

Stepping through a 10 period version of an ABM

To demonstrate the above mentioned effect I take the adjusted balance given by the ABM formula and calculate its interest and principal payment parts. I take a fixed path of property prices which first decline to 80% of the initial value and then to 50% of the initial value to make numbers pop out. We can see the resulting FRM and ABM balance side by side in Figure (4) with all the values and losses calculated. Summing up all the principal payments we see that it's not the total 1.000.000 as with the FRM case. This is because in the ABM definition there is no real balance sheet kept that is decreased with actual principal payments and instead there are only references to the related FRM balance values at the particular time. Figure (4) also shows that we have a ~5% loss on both principal and interest. We can also see how the payments, the balance and the NPV of the total loss of the payments due to payment reductions evolve over time.

ABM				Domestic currency			Falling house price index			
End of Period	Property price	LTV	FRM Balance before	Adjusted Balance before	Principal payment	Interest payment	Total payment	FRM Balance after	Adjusted Balance after	NPV Total Loss
0	1000000	0	0	0	0	0	0	1000000	1000000	
1	1000000	1	1000000	1000000	69029	80000	149029	930971	930971	0.00
2	1000000	0.930971	930971	930971	74552	74478	149029	856419	856419	0.00
3	800000	1	856419	**800000**	75212	64000	139212	775903	724788	2.03
4	800000	0.969878	775903	775903	86957	62072	149029	688945	688945	1.58
5	500000	1	688945	**500000**	68158	40000	108158	595032	431842	5.98
6	500000	1	595032	**500000**	85228	40000	125228	493605	414772	7.35
7	500000	0.987209	493605	493605	109541	39488	149029	384063	384063	6.52
8	800000	0.480079	384063	384063	118304	30725	149029	265759	265759	5.91
9	800000	0.332199	265759	265759	127769	21261	149029	137990	137990	5.44
10	1100000	0.125446	137990	137990	137990	11039	149029	0	0	5.06
Total					952741	463063	1415804			
NPV						342177	949391			
Loss					47259	18661	50609			
Loss %					4.73%	5.17%	5.06%			

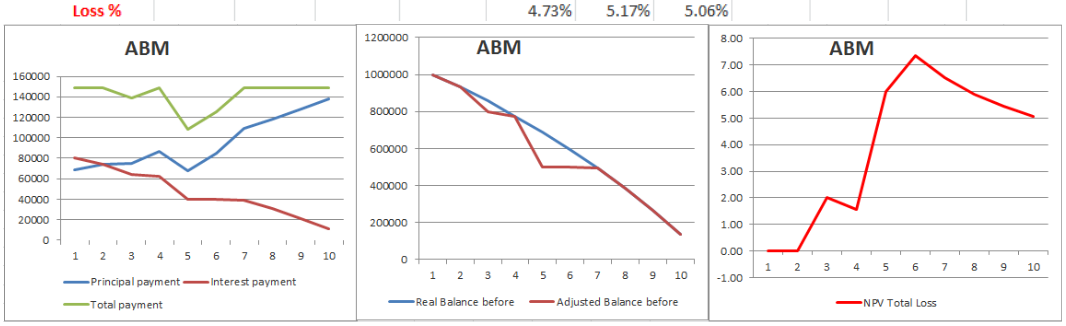

Figure 4: The evolution of a 10 period ABM with payment reductions

3.2 Continuous Workout Mortgages

Continuous Workout Mortgages [17] are similar to the previously discussed ABMs but apply a somewhat more straight forward approach to the same problem. A CWM deals with the scheduled payments directly instead of applying a balance reset from time to time. The main idea is to have payments be directly linked to a house price index no matter whether the borrower has a negative equity or not. This results in payments always being reduced as long as the house price index H_t at time t is less than that of its value at origination H_0. Of course they are only linked to the movement of the house price index as long as the current value is smaller and payments do not increase to be higher than the originally scheduled ones if the cumulative house price index change is positive. The result is a much higher level of protection for the borrower even at times when it is not indicated by the LTV value. Just as with ABMs, payment reductions along the lifespan of the loan form the loss of the lender. This of course results in a supposed interest rate premium to be paid for the embedded optionality. Scheduled payments of a CWM can be expressed as

$$Q_{t+\tau}^{CWM} = min(1, H_{t+\tau}/H_0) Q_t^{FRM} = min(1, H_{t+\tau}/H_0) B_t \frac{r\tau}{1-(1+r\tau)^{-(T-t)/\tau}} \tag{11}$$

where Q_t^{FRM} is the value of the original scheduled value for FRM payments and H_t is the house price index process. The outstanding balance thus takes the following form:

$$B_t^{CWM} = min(1, H_t/H_0) B_t^{FRM}$$

Stepping through a 10 period version of a CWM

In Figure(5) we can see that unlike the ABM, the CWM starts applying payment reductions the first time the property value declines. The LTV never approaches the 1 threshold and remains to be constantly lowered even when the house price recovers. Although the house price drop in this example is extreme on purpose - this shows how the CWM can result in an over-protection and can take bigger losses than needed due to payment reductions.

CWM				Domestic currency			Falling house price index				
End of Period	Property price	LTV	FRM Balance before	Adjusted Balance before	Principal payment	Interest payment	Total payment	FRM Balance after	Adjusted Balance after	NPV Total Loss	Ht/H0
0	1000000	0	0	0	0	0	0	1000000	1000000		1
1	1000000	1	1000000	1000000	69029	80000	149029	930971	930971	0.00	1
2	1000000	0.930971	930971	930971	74552	74478	149029	856419	856419	0.00	1
3	800000	0.856419	856419	685135	64413	54811	119224	775903	620722	6.16	0.8
4	800000	0.775903	775903	620722	69566	49658	119224	688945	551156	9.23	0.8
5	500000	0.688945	688945	344473	46957	27558	74515	595032	297516	16.18	0.5
6	500000	0.595032	595032	297516	50713	23801	74515	493605	246802	20.79	0.5
7	500000	0.493605	493605	246802	54771	19744	74515	384063	192032	24.06	0.5
8	800000	0.384063	384063	307251	94644	24580	119224	265759	212607	23.68	0.8
9	800000	0.265759	265759	212607	102215	17009	119224	137990	110392	23.39	0.8
10	1100000	0.125446	137990	137990	137990	11039	149029	0	0	21.77	1.1
Total					764850	382677	1147527				
NPV						290114	782268				
Loss					235150	70724	217732				
Loss %					23.52%	19.60%	21.77%				

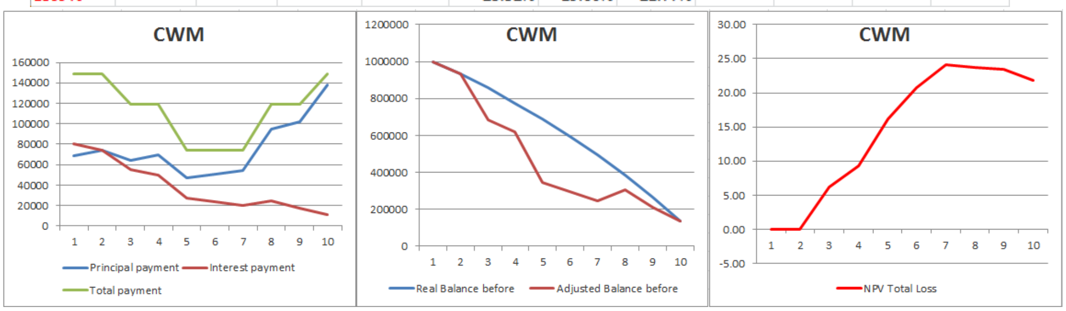

Figure 5: The evolution of a 10 period CWM with payment reductions

4 Ideas for modified AWMs

In this section I present some ideas for modifications of previously presented AWMs that seemed interesting after taking a closer look at how the balance and the payment reductions evolve. I also use this section to give a short example on how both originally proposed AWMs and the ones modified by myself compare to each other on the a above used simple 10 step scenario with both originally scheduled and actually materialized values and losses calculated.

4.1 ABM with No Principal Loss for payment reductions

Instead of always referencing to the original FRM balance we could introduce a real balance sheet to the ABM and decrease this with the actual principal payments done in each period. This way we would only have interest loss but no principal loss. Of course the adjusted balance would be calculated just as the original ABM. The equations for the ABM with real balance decreased by its own principal payments would be the following:

$$Q_{t+\tau}^{ABMNPL} = min(B_t^{Real}, C_{t+\tau})\frac{r\tau}{1-(1+r\tau)^{-(T-t)/\tau}} \tag{12}$$

$$B_t^{ABMNPL} = min(B_t^{Real}, C_t) \tag{13}$$

Here B_t^{Real} is the mortgage's above mentioned real balance which is obtained by calculating periods recursively using interest payment part $I_{t+\tau} = r\tau min(B_t^{Real}, C_{t+\tau})$ which is basically the interest the borrower has to pay on the adjusted balance. Subtracting this from the total payment gives us the Principal Payment part $P_{t+\tau} = Q_{t+\tau}^{ABMNPL} - I_{t+\tau}$. Now the balance after the payment is the balance minus the principal payment $B_{t+\tau}^{Real} = B_t^{Real} - P_{t+\tau}$. This is exactly the same process like the recursive formulas for FRM at Equation (2) where we don't derive closed form equations but keep track of principal and interest payments from period to period.

Using this modified mortgage construction we still achieve the goal of keeping the adjusted balance capped with the estimated price of the property but we have no loss on the principal itself resulting in a ~2.2% loss instead of a ~5% loss.

We can see that the side effect of this is that in the last quarter of the total time we have elevated payment levels. What this modification does beyond the original ABM is that it restructures some of the principal lost at initial payment reductions to the end.

4.2 CWM with payment reductions activated only when LTV>1

As mentioned before, we can see that the CWM has an over-protection as its payment reductions are activated instantly as the property price decreases below the initial value. An idea to correct this over-protection would be to use the LTV itself as the activator of the payment reductions instead of the property price declines. The resulting equations for a CWM with LTV adjusted payments are

$$Q_{t+\tau}^{CWM/LTV} = min\left(1, \frac{1}{LTV_{t+\tau}}\right) Q_{t+\tau}^{FRM}$$

This will differ from the original CWM only in one thing: it activates the payment reductions only at the point when the LTV reaches one. After a closer look and some algebraic manipulations however we can see that in this case $B_t^{CWM/LTV} = B_t^{ABM}$ and $Q_{t+\tau}^{CWM/LTV} = Q_{t+\tau}^{ABM}$ so the LTV activated CWM is exactly the same as the ABM. I moved the proof of the two constructions being

ABM NPL				Domestic currency			Falling house price index			
End of Period	Property price	LTV	Real Balance before	Adjusted Balance before	Principal payment	Interest payment	Total payment	Real Balance after	Adjusted Balance after	NPV Total Loss
0	1000000	0	0	0	0	0	0	1000000	1000000	
1	1000000	1	**1000000**	1000000	69029	80000	149029	930971	930971	0.00
2	1000000	0.930971	**930971**	930971	74552	74478	149029	856419	856419	0.00
3	800000	1	856419	**800000**	75212	64000	139212	781207	781207	2.03
4	800000	0.976509	781207	781207	87552	62497	150048	693655	693655	1.43
5	500000	1	693655	**500000**	68158	40000	108158	625497	500000	5.86
6	500000	1	625497	**500000**	85228	40000	125228	540269	500000	7.24
7	500000	1	540269	**500000**	110960	40000	150960	429309	429309	6.28
8	800000	0.536636	429309	429309	132241	34345	166586	297067	297067	4.58
9	800000	0.371334	297067	297067	142821	23765	166586	154246	154246	3.27
10	1100000	0.140224	154246	154246	154246	12340	166586	0	0	2.23
Total					**1000000**	**471424**	**1471424**			
NPV						**346598**	**977667**			
Loss					0	14240	22333			
Loss %					0.00%	3.95%	2.23%			

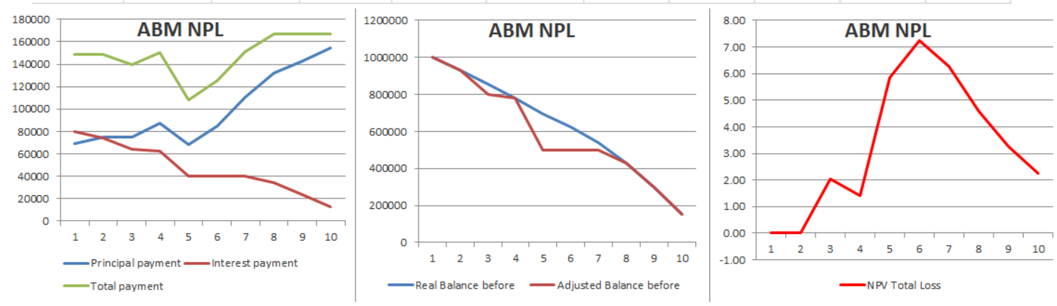

Figure 6: The evolution of 10 period modified ABM with payment reductions

equal from this part of the paper to Appendix (B) to preserve fluency and relocate longer mathematical derivations.

Although with this idea of modifying the CWM I did not end up with an entirely new model it is even more interesting to see that we can view the CWM and the ABM as close but different versions of each other in fact we can view the ABM as an LTV activated CWM.

4.3 Comparing the three models

Because the above result, I only have to compare one extra model to the original two AWMs instead of two. Using the simple scenario from the above examples we can see how each of the model's cumulative NPV of payment losses compare to each other.

Cumulative NPV Total Loss		
CWM	ABM	ABM NPL
0.00	0.00	0.00
0.00	0.00	0.00
6.16	2.03	2.03
9.23	1.58	1.43
16.18	5.98	5.86
20.79	7.35	7.24
24.06	6.52	6.28
23.68	5.91	4.58
23.39	5.44	3.27
21.77	5.06	2.23

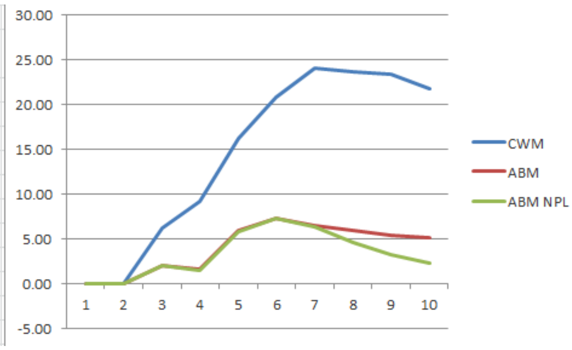

Figure 7: Comparing the three different AWMs

We can see that the two ABM versions are providing decent amounts of payment reductions and keeping the LTV value sub one as seen in the previous tables. The CWM however does end up in overshooting the goal although this example presented is both artificial and extreme on purpose.

5 Quantifying mortgage default risk

The authors proposing the above described two AWMs have made[9, 17] initial proposals for a pricing formula for some of the parameters and have presented sample results in loan valuation. The valuation techniques presented were based on a purely option theoretic approach where the price of embedded options are estimated given the assumption that all borrowers aim to always maximize their expected payoffs of those options. These are the rational prices of the embedded options which are useful in understanding and comparing the differences in the products. It would be interesting to estimate how expected losses would develop given the suboptimal nature of borrower behavior in exercising any embedded option. It is clear that a mortgage is not entered by a borrower just for the sake of searching for optimal times of default and the realized payoffs an average will differ from the optimal as these are not stock options which are traded purely on an expected payoff decision.

The goal of the paper is to take empirical evidence of borrower behavior to build a general mortgage valuation framework that could be used for comparing different kinds of products. For achieving this goal the modeling process was broken down into two important steps:

1. The first step is to create a mathematical model that could explain most of the variation in mortgage default rates seen over the past decade and could re-create most of what has happened just from having given borrower behavior information and the house price index.

2. If the theoretical model developed in the first step does seem to explain most variation in default rates then I can use the model to estimate expected losses of different mortgage constructions and thus proposing a general mortgage valuation framework given empirical evidence of borrower behavior.

5.1 High LTV loss as put option to default

The previously mentioned rationality behind a default can be viewed as the borrower having a put option on its equity which turns in-the-money as $LTV > 1$. A short introduction on options and option pricing can be found in Appendix (A). We can see that by the nature of this optionality we can model this to be similar to an American put option on the house price index.

Why it is similar to an American put option

We could view the default decision as a put option on the equity which can be exercised at any times desired and thus could write the option's payoff function in the form of

$$f_t = (0 - \underbrace{(C_t - B_t)}_{Equity})^+ \tag{14}$$

so we could use the difference of two stochastic processes to transform it to a payoff function on equity level reaching 0. The value of this default optionality already present in an FRM can then be expressed as an American option as

$$\begin{aligned} P_0 &= \sup_{\tau\in\{0..T\}} \mathbb{E}_{\mathbb{Q}}[f_\tau(S_\tau)] = \mathbb{E}_{\mathbb{Q}}[f_{\tau^*}(S_{\tau^*})] \\ &= \sup_{\tau\in\{0..T\}} \mathbb{E}_{\mathbb{Q}}[(B_\tau - C_\tau)^+] = \mathbb{E}_{\mathbb{Q}}[(B_{\tau^*} - C_{\tau^*})^+] \end{aligned} \tag{15}$$

where τ^* is the optimal stopping time and $\mathbb{Q}$ is the Equivalent Martingale Measure. See Appendix (A) for details on option pricing.

We can see that the value of this put option is the expected high LTV induced loss of the lender - more precisely the expected principal loss. By calculating the value of this put option we have the estimate for the high LTV branch of equation (29), namely the value of $\mathbb{P}(LTV > 1)P(D|LTV > 1)\mathbb{E}[Loss_D|LTV > 1, D]$. The value of the option could be calculated using American option pricing methods (e.g. Longstaff-Schwartz) which require a good model for the house price index process H_t or using purely empirical pricing methods with historical values of a given house price index.

Why it is different from an American put option

The previous argument would give us the price of such an option by itself but as mentioned in the beginning of this chapter this will not give us the expected losses as this would need a behavior from borrowers to constantly search for the optimal time τ^* of defaulting to maximize the expected loss of the bank and maximize their payoff function of the default. As mentioned at the beginning of the chapter - I aim to create a mortgage valuation framework that would incorporate user behavior affected by macroeconomical processes and to measure the expected loss given the suboptimal exercise of default options. In the next sections I will give more detail on how to bring suboptimal lender behavior and options together.

5.2 Types of risks involved in mortgage products

To compare risks in different mortgage products I take the baseline to be a traditional (FRM) mortgage with the best scenario possible, namely payments happening regularly according to contract and no types of the following risks are being realized: no default (no principal loss and no missed further interest payments), no prepayments. I will deal with all of the risks previously listed except prepayments.

More complex mortgage products that try to minimize default risk have some kind of payment reduction built in that forms the loss of the lender when active. I will deal with all types of risks separately and call **"Total Cost of Realized Risks**" the total cost of suboptimal events, which might be either losses or unrealized profits, namely the cost of all negative events compared to the optimal scenario. The importance of adding up the costs of all realized risks is to get an "expected difference from the optima" comparison of different types of mortgages. The structure of Total Cost of Realized Risks is:

- default induced events
    - principal loss (*Loss*)
    - missed further interest payments after default (I_{missed})
- payment reduction cost (*PRC*)

I want to focus on default related risks in the paper and to omit prepayment as possible which is possible in this case given that I assume the bank is effective in reallocating capital and can reallocate the prepaid capital to initiate new mortgages and thus the future interest payments are not yet lost but will have to come from new loans and different borrowers serviced by the capital received back as prepayments. This efficiency assumption helps to avoid modeling prepayments in default related risks.

I will compare different mortgage products by the expected value of their Total Cost of Realized Risks. Note that the different types of risks are not estimated independently but realizations of risks of mortgage scenarios are broken down to these components.

5.2.1 Principal loss caused by default

The main idea for modeling default induced losses is to break up the probability space into regions that would suit the option theoretic nature of the underlying decisions. The details of how I did this was moved to Appendix (C) to preserve fluency and to relocate this longer mathematical argument. The resulting core building block of estimating default related losses is the following:

$$\begin{aligned}\mathbb{E}[Loss] &= \underbrace{\mathbb{P}(LTV \leq 1)P(D|LTV \leq 1)\mathbb{E}[Loss|LTV \leq 1, D]}_{low\,LTV\,default\,loss} \\ &\quad + \underbrace{\mathbb{P}(LTV > 1)P(D|LTV > 1)\mathbb{E}[Loss|LTV > 1, D]}_{high\,LTV\,default\,loss} \end{aligned} \tag{16}$$

Using this segmentation of the probability space I can start with the influence of house price changes on the LTV. Given that it is either rational or irrational to default I can estimate borrower behavior induced probabilities.

The low LTV branch is not omitted as it is not zero due to the fact that the above value for loss needs to contain the part of the expensive foreclosure costs that may not be covered fully by the collateral.

5.2.2 Missed interest payments after default

Given that a default event occurs at time t, the missed remaining interest payments are

$$I_t^T = \sum_{k=t}^{T} I_k \tag{17}$$

where I_k is the interest payment scheduled for time k and I_t^T is the cumulative interest payments scheduled from time t of the default to maturity.

5.2.3 Payment reduction cost

The costs of offering payment reductions to the borrowers depend on both the type of mortgage product and the type of payment reduction included. As all mortgage products have their monthly payments specified, the cost of payment reductions for an AWM is

$$PRC = \sum_{t=1}^{T} Q_t^{FRM} - Q_t^{AWM} \tag{18}$$

Of course we do not know the future values of the terms Q_t^{AWM} as it is path dependent (it will highly depend on the initial LTV of the borrower and the future of the house prices).

5.3 Fitting the option theoretic model and the expected loss model together

When pricing a conventional equity option we have the completely natural assumption that the holder of the option will "act rationally" and exercise the option when he expects to maximize the resulting return. The default optionality in a mortgage is not the typical case of options so we need to closely examine any assumption related to exercising. I will use notation that fits options in general to give a clear analysis of the option itself without mortgage related concepts appearing. We can segment the probability space of the option payoffs similarly to that of the principal loss

$$\mathbb{E}[Payoff] = \mathbb{P}(ITM)\mathbb{E}[f_t|ITM] + \underbrace{\mathbb{P}(OTM)\mathbb{E}[f_t|OTM]}_{0}$$

where ITM means **In-the-money** and OTM means **Out-of-the-money**. It is important to notice that we can stop here only if dealing with a conventional equity option because of the assumption of exercising when it is most profitable which we can formulate as an implicit $P(Exercise|ITM) = 1$ term being present. This implicit assumption will change for the case of the default optionality and will have a probability different than 1. After discarding the zero valued OTM term the full segmentation of the probability space is as follows:

$$\mathbb{E}[Payoff] = \mathbb{P}(ITM)\mathbb{P}(E|ITM)\mathbb{E}[f_t|ITM,E] \tag{19}$$

where E denotes the event of exercising.

We can see that by just doing a regular option pricing with the conventional assumption of always exercising when it is most worth it we would get the **value of the option** but not the **expected payoff**, as the latter will also include a term describing the behavior of the holder of not always taking advantage of the option. This breakdown of the expected payoff fits the model of the principal loss, where by now it is trivial that ITM will be the case of $LTV > 1$ and exercising is the event of the default. Figure (8) explains visually how the two models fit together. The expected payoff and the top branches of the expected losses still differ since the expected payoff does not include foreclosure costs. Losses due to foreclosure costs will be present on the lower branches where $LTV \leq 1$. After taking into account both borrower behavior specific exercise probabilities and adding foreclosures we have fully covered all possible losses.

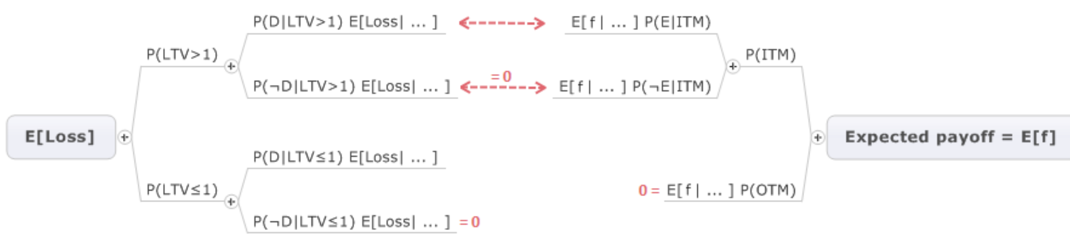

Figure 8: Fitting the default option model to the expected loss model

5.4 More sophisticated versions

After presenting the main structure of capturing losses driven by borrower behavior I have to note that these branches further expand at every time step and house price values together with borrower decisions will choose a certain path along a longer probability tree as seen in Figure (9). Every time step consists of two events. In the first event the house price and the previous balance determines the actual LTV independently of the borrower. The borrower's probability of default in a month is determined by the conditional distribution $\mathbb{P}(D|LTV_t = x)$ after observing this LTV value.

Given a good dataset containing LTV information it is possible to estimate a joint distribution of the defaults and the LTV values. Of course the distribution of the LTV would not be representative for a general case as house prices have many year long trends and a data set would capture only realizations of an interval of time but I assume the conditional probabilities $\mathbb{P}(D|LTV_t = x)$ to be fairly stable as they capture the essence of the behavior and are the primary interest for my model as shown in figure (9). To estimate these conditional probabilities used binning/discretization of the distribution and measured bands of the form $\mathbb{P}(D|x_1 \leq LTV_t \leq x2)$ where I took the bands to be 0.1 wide and estimated 20 conditional probabilities in the LTV range of $[0,2]$. Every active month of a loan contributes to an example of being active at that particular LTV and a default trivially equals an example of a default decision at that LTV. From this we can use simple relative frequencies to estimate the conditional probabilities.

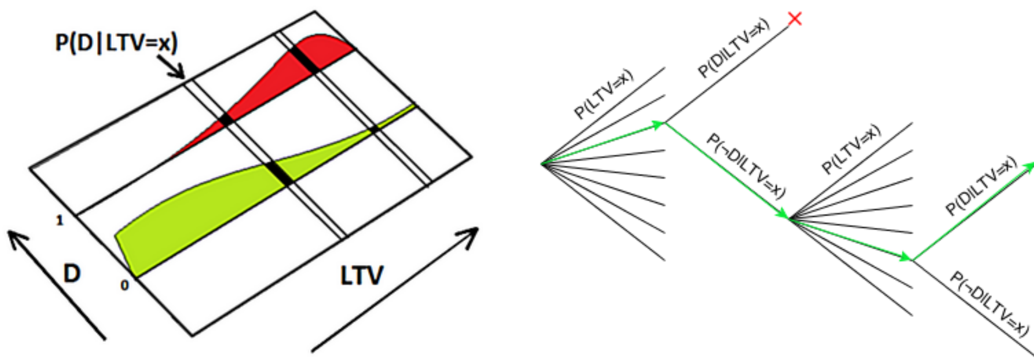

Figure 9: Sketch of D and LTV joint distribution along with the evolution of a loan

In this model I view defaults as the first arrival in a more complex Bernoulli process where we have conditional probabilities for the Bernoulli that are driven by the realizations of the most current LTV. This was also the basis for running simulations.

5.5 Comparison of mortgage products using the above model

Having calculated the expected costs of different types of risks involved in different mortgage products I take the **Total Cost of Realized Risks** to be the sum the costs of individual risk types

$$TCRR = Loss + I_{missed} + PRC \tag{20}$$

This represents the total difference compared to having an FRM mortgage under an optimal scenario of no defaults or any other kind of risk realized. If two mortgage products have the same $\mathbb{E}[TCRR]$ value then they could be priced to have the same interest rates. Given a complicated mortgage product with built in payment reductions we would need to price an interest rate premium on the difference of expected “costs” involved so the interest rate premium on top of a traditional FRM interest rate will be based on the $\mathbb{E}[TCRR(AWM)] - \mathbb{E}[TCRR(FRM)]$ value.

6 Simulation and results

This section presents the data used to test the model developed in the previous section along with some measurements, model validation and mortgage pricing simulation results.

6.1 Data used for statistics and simulation

Loan level data

The dataset I used for initial statistics and simulations was the loan-level credit performance data released by Freddie Mac called Single Family Loan-Level Dataset. The dataset includes loan-level origination and monthly loan performance data on a portion of the fully amortizing 30-year fixed-rate single-family mortgages acquired by Freddie Mac. Only those loans were put in the dataset which were labeled as "full documentation". After short research on historical default rates I chose to work with all loans provided which were originated in Florida from 2000 to 2010 as it had one of the highest default rates and highest variation or default rates between years and was perfect for further analysis.

The term "Default" used throughout the paper was used as a general term for a variety of loan early termination categories. The loans that were labeled as defaulted loans as seen in Figure (10) ended either in a Foreclosure procedure, a Short sale in which the property is sold by a financially distressed borrower for less than the loan amount, a Deeds-in-lieu in which the borrower also decided to walk away from the loan and gave back the house to the bank, or seriously delinquent loans that were sold with loss by the bank to third parties. Figure (11) shows that defaults were not happening with the same number of months after origination in different years but most of them happened withing the same 3 year period if there were enough active at that time no matter the origination date.

House Price Index and price distribution

For the change in house prices I used the state level Freddie Mac House Price Index (FMHPI) along with the Federal Housing Finance Agency's state level measurements on housing price volatility around the aggregate mean. The advantages for using the FMHPI instead of the more widely used S&P Case-Shiller Home Price index is that the FMHPI provides monthly index values from 1975 instead of just quarterly ones like the previous. To generate a realistic real estate price

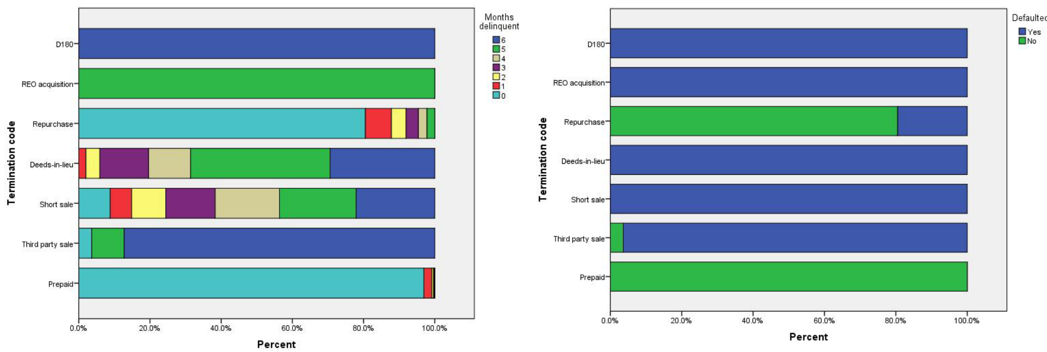

Figure 10: Early terminations and my definition of the default

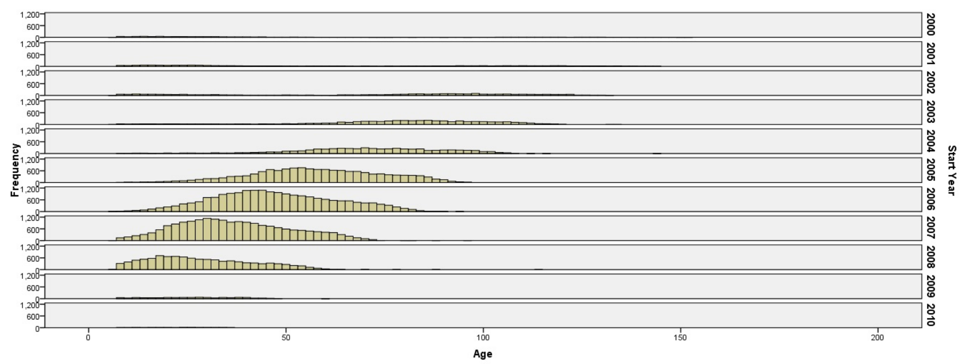

Figure 11: Defaulted loans from different origination years

distribution I used the FHFA's proposal of parameters to generate a Geometric Brownian Motion around the aggregate mean house price index to model individual house price movements as shown in Figure (12). Each house is a separate trajectory and all calculations of the model and the simulations were made to reflect reality as much as possible: mortgage products all used the estimated house price value by just observing the initial house value and the house price index while individual borrowers knew more about their houses and were assumed to always know the real market price of their property. Of course all of the house's random movements around the mean index start at their own respective mortgage's origination date and property value as known precisely due to the mandatory appraisal and so Figure (12) is just an illustrative example for the possible paths for one home. Volatilities were scaled up just for the sake of creating these sample plots.

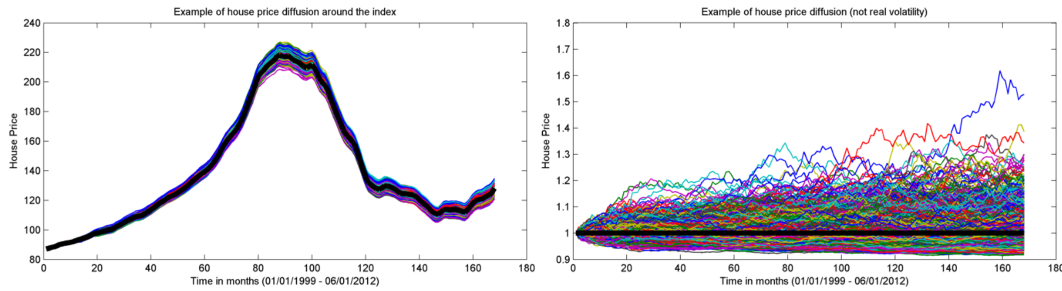

Figure 12: House price index and trajectories of individual houses

6.2 Statistics and measurements for the model's parameters

I examined the data grouped by the loan's age of origination. As Figure (13) shows, there is a huge variation in Cumulative Default Rates (CDR) after origination among loan cohorts of different origination age. After 2003 default rates started rising every year until its peak of 2007. An interesting observation is that the CDR of year 2008 starts growing faster but ends up slowing down in the consecutive months and CDRs after and including 2009 are back to the pre-2003 years.

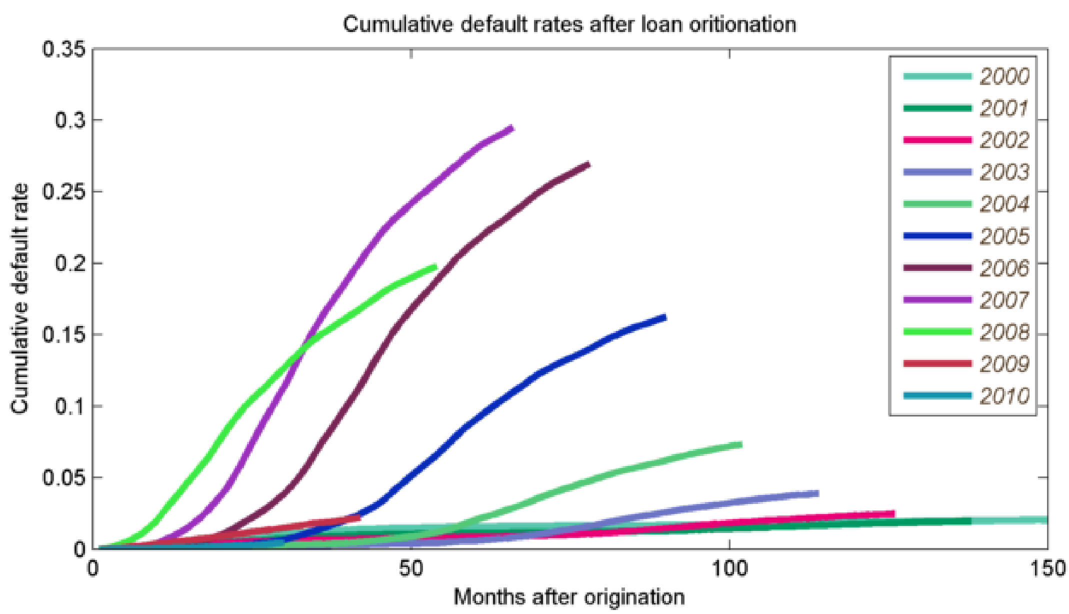

Figure 13: Cumulative default rates, months after origination

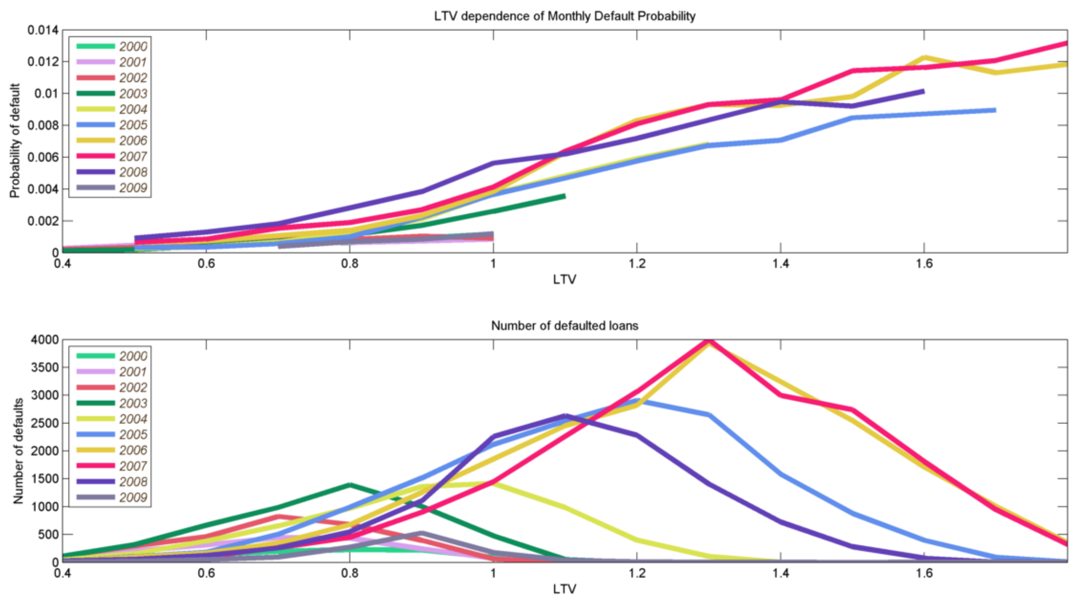

Figure 14: Monthly probability of default for LTV values

To assess this behavior I calculated statistics for the conditional probabilities $\mathbb{P}(D|LTV = x)$ discretized in 10% LTV bands. As expected, LTV dependent default probabilities remain fairly stable throughout the years with more error on regions where less data was available. Aside from some small variances at low sample size areas, the $\mathbb{P}(D|LTV = x)$ seems like a monotonically increasing function of the LTV and quite stable given the huge variations in yearly default behavior between 2000 and 2010. It is interesting to note that up until 2004 these conditional probabilities were lower just under the $LTV = 1$ threshold and the biggest variation can be seen at this point. We can also note that default probabilities around $LTV \approx 1$ were slightly increasing over time and were more stable in areas either much below or much above one but we can also see some deterministic movements.

After looking at unemployment rates between 2000 and 2009 as seen in Figure(15) I concluded that incorporating it would not explain small changed in default probabilities as unemployment rates were declining until 2007 whereas the increase of default probabilities around $LTV \approx 1$ started to increase much before that.

Figure 15: Unemployment rate, Florida[20]

6.3 Backtesting the model on real data

At the beginning of chapter (5) I stated that it is important to test whether or not the constructed model can explain most of the variation in default rates in the data so that it could be used to price mortgages constructions.

I used the following approach. Given the model constructed in the previous chapter with estimated conditional probabilities and a house price index trajectory, the model should give cumulative default rates close to that of the real data by simulating a 30 year FRM mortgage on the the historical house price index. For this of course it would mean nothing if for each year I would use the conditional probabilities estimated from that year because I would just reproduce the same thing that I estimated on. To test how well the model captures the borrower's behavior I used only one single set of conditional probabilities for reproducing cumulative default rates of all years between 2000 and 2010.

After this initial test run I got close results to that of Schelkle's second examined model[8]. He presented models different than the one than I have developed so I will give a short summary of them for the sake of comparison. The first of the models presented was a purely option theoretic threshold model where borrowers defaulted when reaching a given LTV threshold with the parameter needed to be estimated via grid search for best fit instead of measuring probabilities that drive behavior. He concluded that the model did not fit the data well at all. The second model he presented was a so called "double trigger" model in which borrowers with $LTV < 1$ could not default and borrowers with $LTV > 1$defaulted each month with a given fixed probability that also needed to be estimated for best fit. He concluded that this model is successful in explaining most of the variance in cumulative default rates of individual year. My first results of my model were similar to that of his.

To further improve the accuracy of the model, after concluding that unemployment rate would not be related to the slight probability shifts early on, I added a trend anticipating effect to the model where the borrowers would scale their likelihood of defaulting with anticipating a trend in the housing market given by the total ratio of change $\frac{H_t}{H_{t-n\tau}}$ over the past n periods. Finding the number of months for long term look-back did not require sophisticated optimization algorithms as there was only a few discrete number of values to check and hence the best fit could be done in a grid search fashion. The best fit for trend anticipation was $n = 14$ which assumes a trend anticipation with around a one year look-back (plus the lag or probably viewing only a quarterly released house price index).

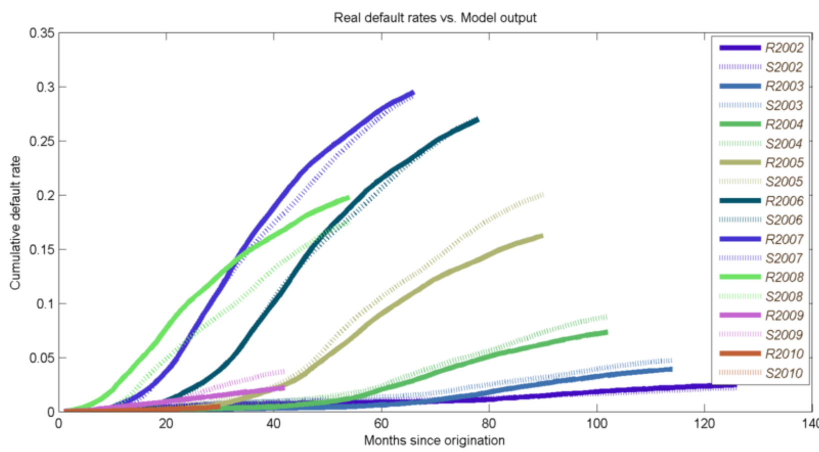

Figure 16: Cumulative default rates, original vs simulated given House Price Index

After incorporating a trend anticipation in user behavior the model got promising results for reproducing the changes in default rates between years 2000-2010 as seen in Figure (16). I conclude that the model explains most of the change in default rates in the given years and managed to fit reality even better than results in [8].

6.4 Comparison and pricing

For pricing AWMs I compared their Total Cost of Realized Risks as presented at section (5.5). I will use the concept of the Internal Rate of Return to calculate what effective returns does the expected cash flow result for each mortgage construction. In general, the price of such a mortgage should be a function of the borrower's starting LTV and the house price index model we assume to drive property prices.

Modeling house price indices is a complicated and yet unsolved problem. Many papers in the literature use a simple Geometric Brownian Motion with estimated drift and volatility parameters as there's no de facto standard for a house price index model to use for Monte Carlo simulation. The problem of the model is really due to the fact that volatility itself doesn't contribute too much in the house price process but instead it's clear that we have trends that stick for a number of years and change whenever something in the fundamentals change. What makes house price processes more complicated is that these trends are mostly due to current government plans for loan regulations, interest rates and many other factors. Just to see the results I also checked whether or not the log differences of the house price index process are a normally distributed. The one sample Kolmogorov-Smirnov test gave significance 0.000 and thus rejected the hypothesis of the distribution being normal as seen on Figure (17).

Hypothesis Test Summary

	Null Hypothesis	Test	Sig.	Decision
1	The distribution of DIFF(log_fl_2012, 1) is normal with mean 0.00 and standard deviation 0.01.	One-Sample Kolmogorov-Smirnov Test	.000	Reject the null hypothesis.

Asymptotic significances are displayed. The significance level is .05.

Figure 17: Testing normality of the log house price increments

To use a realistic house price distribution I used bootstrap sampling from the house price indices provided from all 381 Metropolitan Statistical Areas and thus using a fully empirical approach for the house price process. After running a simulation for different parameter values I obtain expected values of the cash flow on the full mortgage time horizon and solve the following equation for IRR to get the Internal Rate of Return of the expected cash flow:

$$-B_0 + \sum_{i=1}^{T/\tau} \frac{Q - \mathbb{E}(TCRR_i)}{(1 + IRR\tau)^i} = 0 \tag{21}$$

I did the same procedure with all four mortgage constructions with exactly the same parameters. In general, for given common mortgage parameters (foreclosure cost fc, starting loan-to-value ratio LTV_0,reference interest rate r, maturity T, compounding period τ) I give the resulting internal rates or return for each mortgage construction

$$v(fc, LTV_0, r, T, \tau) \rightarrow \{IRR_{FRM}, IRR_{CWM}, IRR_{ABM}, IRR_{ABMNPL}\}$$

which are directly comparable. It is interesting by itself how much lower the actual IRR of a mortgage construction is of its reference interest rate but it is much more interesting to see how they compare to each other.

6.4.1 Sensitivity to starting LTV

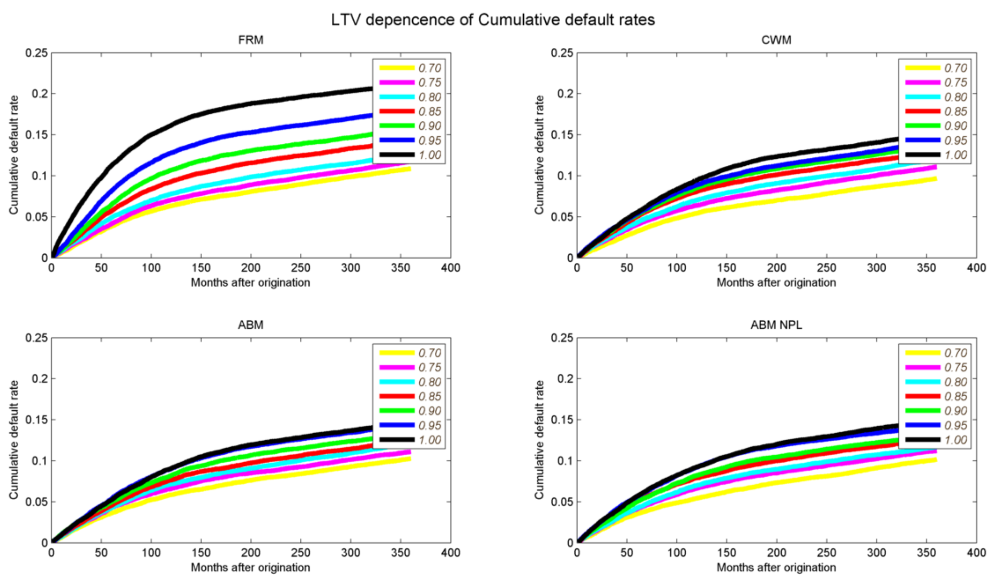

Figure 18: Cumulative default rates for different LTVs

We can see in Figure(18) that the FRM is most sensitive on different starting LTV values for cumulative default rates while all AWMs gave very similar results.

The dependence of the principal loss on the starting LTV in Figure(19) shows similar results to that of the default rates but we can see the modified ABM NPL to recover some of its loss in later periods.

The Interest shortfall in Figure(20) shows that the FRM is the most LTV sensitive however CWMs underperform almost everything under $LTV < 0.85$ due to lowering scheduled payments even when it is not motivated by a high LTV.

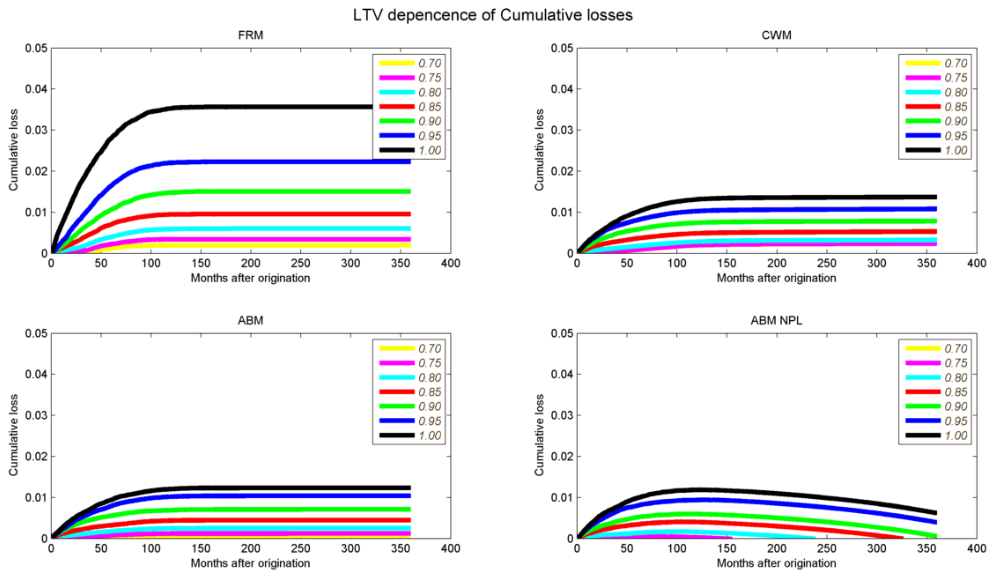

Figure 19: Cumulative principal losses for different LTVs

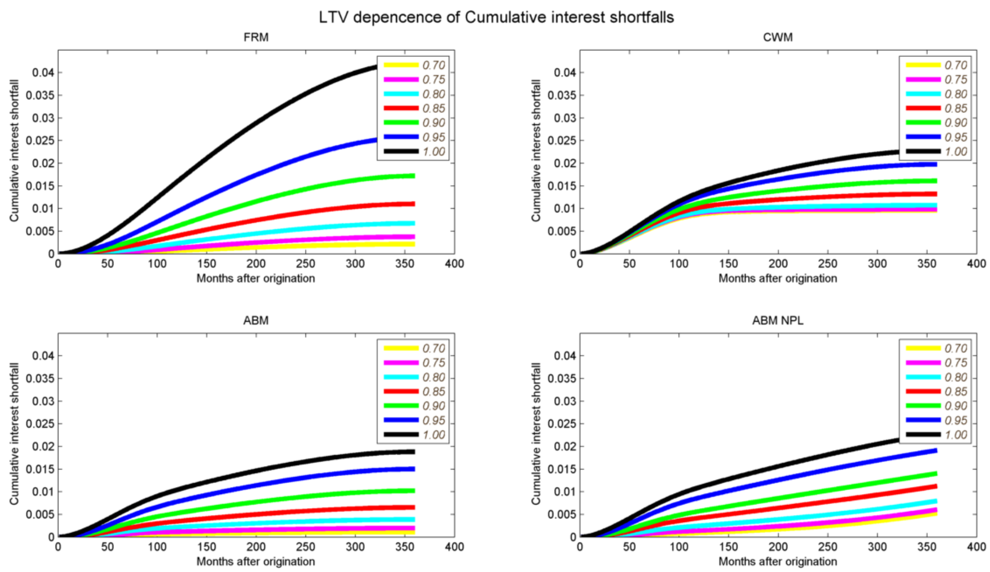

Figure 20: Cumulative interest shortfalls for different LTVs

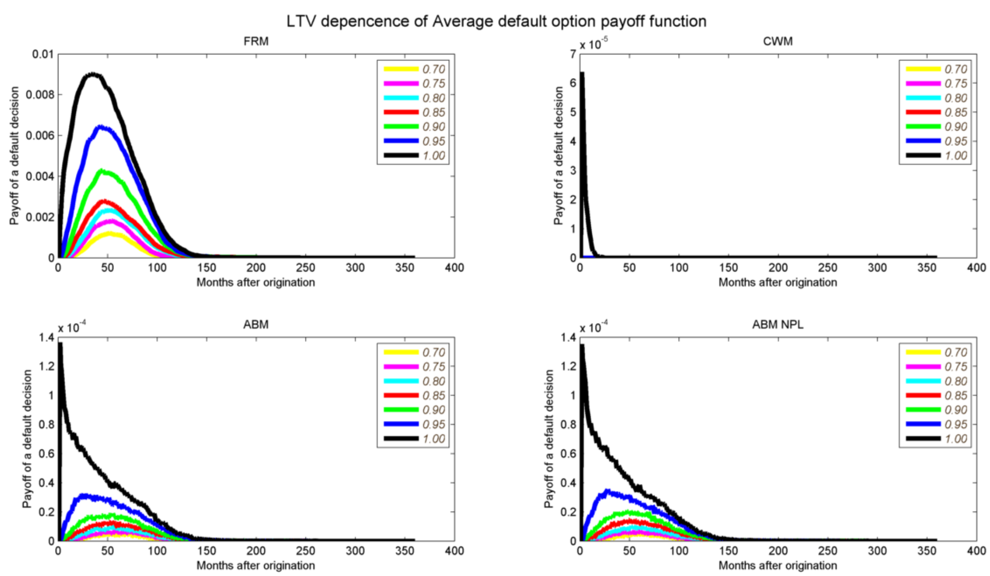

Figure 21: Average default option payoff function for different LTVs

What is most interesting to see is the average option payoff functions in Figures (21,22). The default option of the FRM is in-the-money on the average and it is very LTV sensitive in the first third of the contract, in this case for approximately the first 10 years. The two types of ABMs have identical payoff functions and are by two magnitudes lower on average and also decreasing rapidly. This can be seen best on Figure(22) as it presents the different mortgage types on the same plot zoomed in. The CWM's default option can almost never be in the money except when we start the loan from an $LTV = 1$ critical value. It is important to note that the only reason AWMs here can have positive default option value is due to simulating a realistic house price distribution around the mean with the diffusion of individual house prices. This results that as we expect, the bank's estimate of the LTV is very accurate but not perfect and hence some loans will have bigger or smaller real LTV values than estimated.

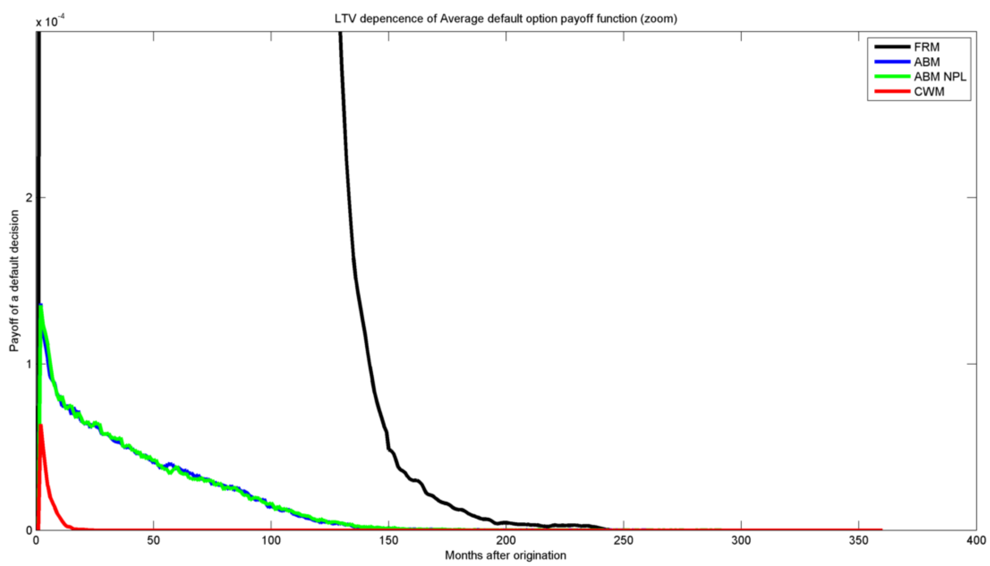

Figure 22: Default option value(zoom)

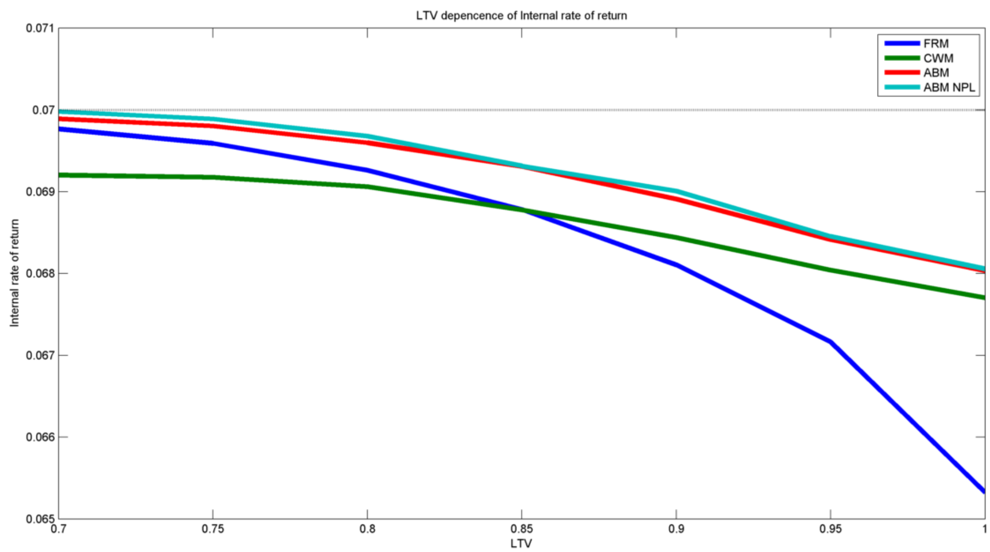

Figure 23: Internal return rates for different LTVs

As anticipated, the FRM is very sensitive to starting LTV values and this affects all aspects of the loan from default rates, principal losses, interest shortfalls and In-The-Money default option prices. All of these of course will result that its internal rate of return will be one of the lowest. As seen in Figure(23) ABMs give a better return on the same capital and are considered better investments both in low and in high starting LTV situations. The CWM however is prone to overprotection and is not wort using under $LTV \leq 0.85$ as it underperforms all constructions. My proposed idea of a modified ABM with no principal losses due to payment reduction slightly outperforms the simple ABM but the small difference might not be worth the hassle of slightly rising scheduled payments after the 20 year mark although I checked that the slightly elevated scheduled payments would mostly remain lower than starting values if adjusted by inflation. It might be surprising that realizing the FRM also has a “built in” option on top of the FRM interest rate because the ABM will outperform the FRM. As seen in the above examples, the ABM’s payment reductions could be a small price to pay for much bigger gains. The origination of a CWM however is not so simple as it underperforms the FRM in lower LTVs than 0.85 and thus needs to be charged the difference in expected return.

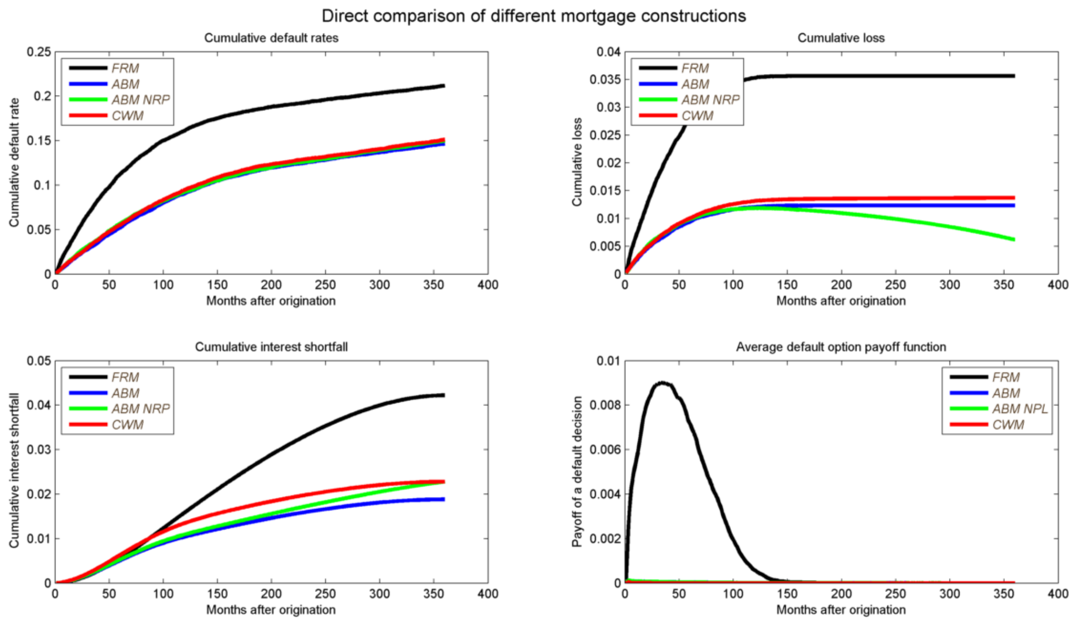

Figure 24: Direct comparison of AWMs

A direct comparison of different constructions is available in Figure(24). We can see that on average, AWMs result in less defaults, losses and interest payments and the biggest difference is not which AWM we use but whether or not we use one or stick to the FRM.

6.4.2 Performance of AWMs on the 2000-2010 house price index

We've already seen that AWMs work well in expectation when sharp house price turns are relatively rare. Because the goal of these constructions are to work even in more extreme cases I also check the performance of the different types of constructions on the house price index and starting CLTV values from the Freddie Mac dataset to see how we would expect them to behave under similar extreme circumstances. Figure (25) shows that although we do see some increase of cumulative default rates (note that defaults increased even in low LTV cases in the data) AWMs would seem to do their job at keeping defaults at the minimum even in "worst case" situations.

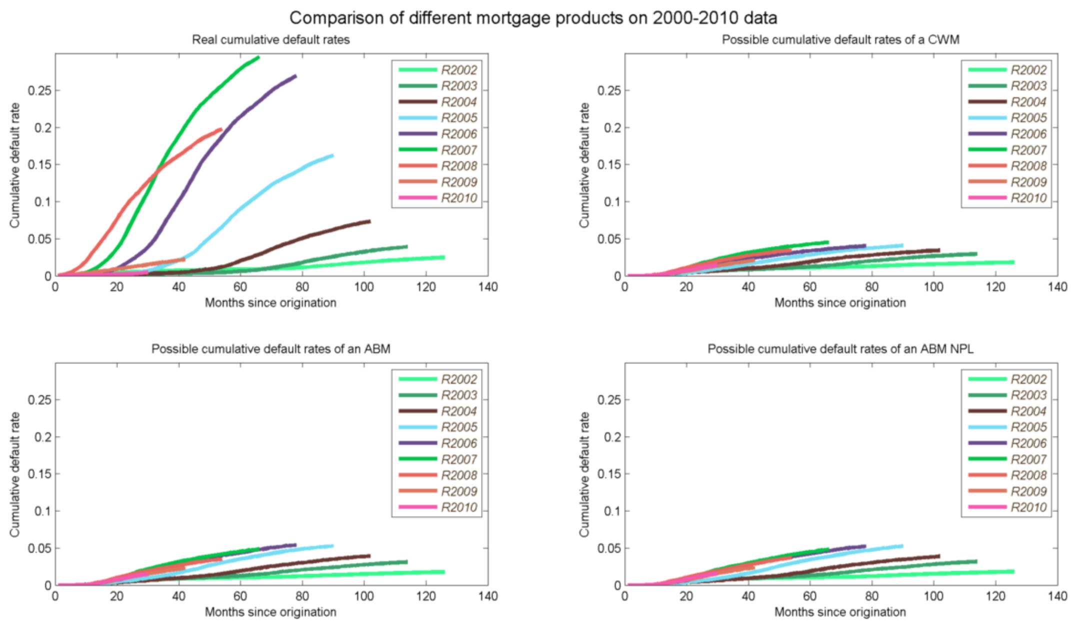

Figure 25: AWMs estimated default rates under real HPI and CLTV data

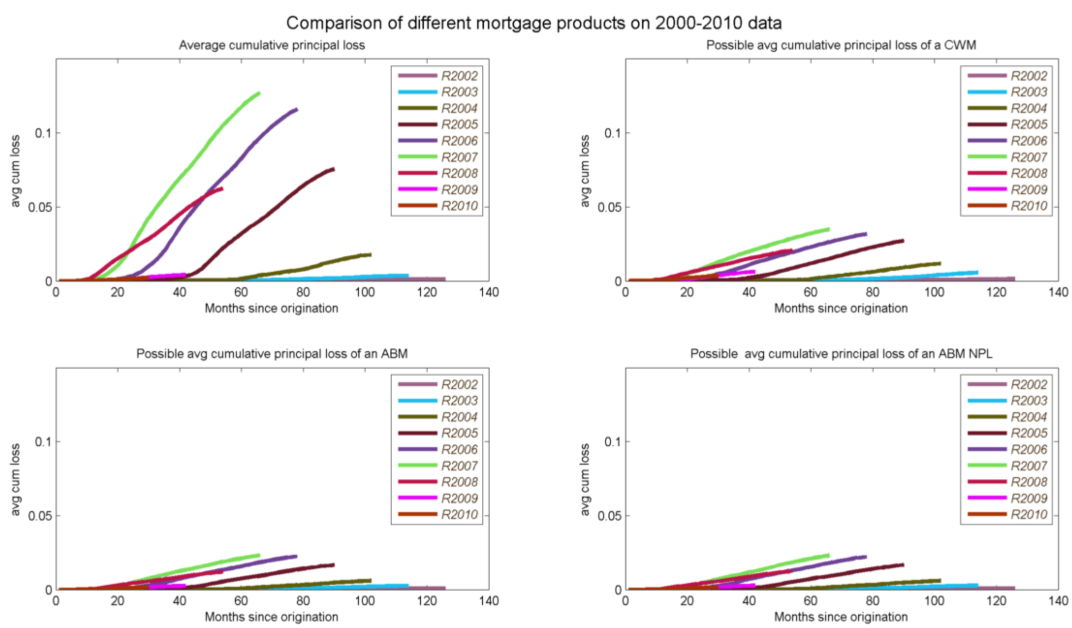

Figure 26: AWMs estimated principal losses under real HPI and CLTV data

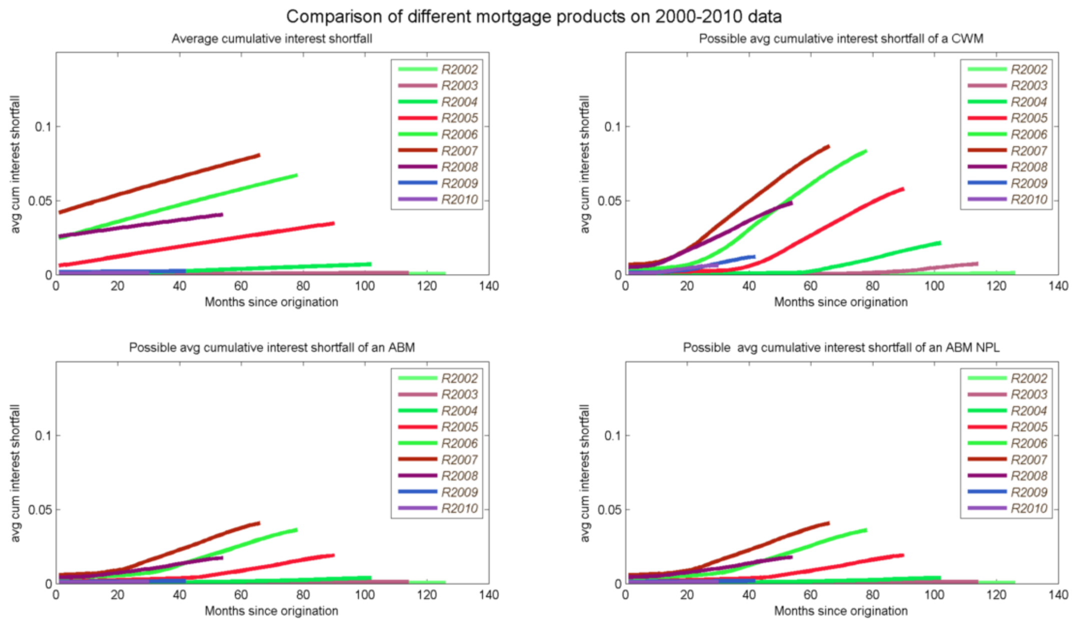

Figure 27: AWMs estimated interest shortfalls under real HPI and CLTV data

Keeping cumulative default rates low is a good result but they have to pay a price for it. Figure (26) shows that average principal losses were quite substantial but still much lower than in the FRM case. Figure (27) shows that the biggest price they paid was sacrificing most of the interest payments which were even more than the FRM in the case of the CWM but much lower in the cases of the ABMs. Although nothing's for free, the performance of AWMs are quite impressive in these extreme circumstances but not because of keeping default rates low - they were designed for that specific purpose - but instead that they do that with much less overall losses than the FRM had to go under.

6.5 Possible direction to continue

Although most tests for most years fit well on the data; 2005 being somewhat of a tipping point in behavioral probability changes and was less accurate as well as 2008 where unemployments really first struck the masses and was also the point of the mortgage market crisis becoming a global financial one. A possible further step would be to incorporate unemployment rates into the model as a probability of losing liquidity.

7 Conclusion

Proposing new or innovative ideas in periods of financial crisis is not uncommon. In fact before we are hit by something unexpected we don't yet know that we need protection from it. The 30-year old fixed rate mortgage that we now call traditional was the result of the efforts of the Federal Housing Administration after the Great Depression. Such events remind us that the need for financial innovation is not less after a crisis but instead higher than than ever.

In the previous decades we have seen the completely legal approaches for handling defaults and to make the choice of the financially rational decision under a negative equity costly and painful in order to reduce their frequencies as much as possible. A global financial crisis could often lead to stop and rethink some of the ideas that were previously enforced without analyzing whether or not there is a better option. The new approaches in constructing mortgage products look irrational at first glance because of their willingness to forgive parts of payments automatically, built into the contract. As there were many proposals on how to treat troubled macroeconomic situations with innovative mortgage products and how to value the rational exercise of their possibilities; I have yet to discover a paper developing a general framework for incorporating empirical evidence of sub-rational behavior to price such products against each other instead of an abstract unreachable reference value. In my attempt to create such a framework I was also surprised how effective these mortgage products can be.

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A Options and option pricing

I will give a short introduction to financial options below to make the concept of embedded optionalities in the mortgage contract clearer and to later highlight the slight differences between pricing regular stock options and the mortgage valuation model I propose.

Options are derivatives of the underlying asset that promise some payoff specified as a function of the asset. There are many types of options but in this short introduction I will only cover option types that are useful in understanding the option theoretic model developed for pricing mortgages.

Before going further I'll define a useful notation that is commonly used while working with options:

$$x^+ = \begin{cases} x & if\, x \geq 0 \\ 0 & if\, x < 0 \end{cases} \tag{22}$$

A.1 European options

One of the most common and most simple types of financial options are European style options. I will present the European Put option which has the payoff function:

$$f(\omega) = (K - S_T(\omega))^+ \tag{23}$$

where S_T is the price of the underlying asset at time T and K is the strike price we agree on. If the price of the asset at the expiration is more than K then the option is worthless and pays no money. On the other hand if the price of the asset at expiration is less than K the option pays the difference, whatever the price of the stock may be. The value of the option at maturity can be viewed on Figure (28). The option is called In-The-Money (ITM) if having a payoff greater than zero, Out-of-The-Money (OTM) on the opposite side resulting in zero payoff and At-The-Money (ATM) at the strike price.

The price of this option can be given by a function of the form $P(S_0, K, T, r, \sigma)$ with the following parameters:

- S_0 - the current price of the asset
- K - strike price
- T - expiration of the option

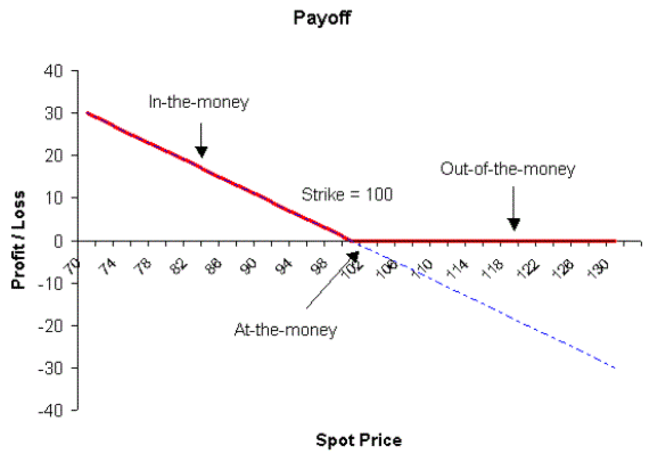

Figure 28: European Put option payoff function[12]

- σ - volatility
- r - interest rate

It is important to note that the European option types can not be exercised before its expiration time T.

A.2 American style options

The only difference between a European option and its American counterpart is that the latter can be exercised at any time desired on the full $0..T$ time horizon and pays the value given by the option payoff function that that time and thus for a simple American put option we have a payoff function

$$f_t = (K - S_t)^+,\ t \in (0, T] \tag{24}$$

A.3 Option pricing

European options

The price of an option in general is the expectation of the discounted claim on the payoff function with respect to a distribution Q that makes the underlying process fulfill the martingale property and is named the Equivalent Martingale Measure (EMM) The value of a European Put option can be expressed as:

$$P_0 = \mathbb{E}_{\mathbb{Q}}\left[e^{-rT} f(S_t)\right] = \mathbb{E}_{\mathbb{Q}}\left[e^{-rT}(K - S_T)^+\right] \tag{25}$$

The price of European options can be given with a closed formula[13].

American options

Because these options can be exercised at any time up to expiration the above equation for pricing a European Put option will be modified as

$$P_0 = \mathbb{E}_{\mathbb{Q}}[f_{\tau^*}(S_{\tau^*})] \tag{26}$$

where $\tau^*$is called the optimal stopping time. The rationality behind such an option therefore is to maximize the expected payoff by exercising when we expect it to take on its highest value. The valuation of such an option is complicated and a closed formula for pricing does not exist although there are several numerical algorithms that can be used[14, 15].

B Proof, the proposed CWM/LTV is the same as the ABM

Proof.

$$\begin{aligned}
Q_{t+\tau}^{CWM/LTV} &= min\left(1, \frac{1}{LTV_{t+\tau}}\right) Q_{t+\tau}^{FRM} \\
&= min\left(1, \frac{C_{t+\tau}}{B_t^{FRM}}\right) Q_{t+\tau}^{FRM} \\
&= min\left(1, \frac{C_0 \frac{H_{t+\tau}}{H_0}}{B_t^{FRM}}\right) Q_{t+\tau}^{FRM} \\
&= min\left(1, \frac{C_0}{B_t^{FRM}} \frac{H_{t+\tau}}{H_0}\right) B_t^{FRM} \frac{r\tau}{1-(1+r\tau)^{-(T-t)/\tau}} \\
&= min\left(B_t^{FRM}, \hat{C}_{t+\tau}\right) \frac{r\tau}{1-(1+r\tau)^{-(T-t)/\tau}} \\
&= Q_{t+\tau}^{ABM}
\end{aligned}$$

$$\begin{aligned}
B_t^{CWM/LTV} &= min\left(1, \frac{1}{LTV_{t+\tau}}\right) B_t^{FRM} \\
&= \begin{cases} B_t & , if\, LTV \leq 1 \\ \frac{C_t}{B_t^{FRM}} B_t^{FRM} & , if\, LTV > 1 \end{cases} \\
&= \begin{cases} B_t & , if\, LTV \leq 1 \\ C_t & , if\, LTV > 1 \end{cases} \\
&= B_t^{ABM}
\end{aligned}$$

■

C Details in the steps of developing the model for default loss

In this section I present the different approaches I tried to model default related losses. I present the initial version also because it directly compares to the model used in the Basel II framework.

The idea of the Basel II framework

The default loss on a traditional fully amortizing Fixed interest Rate Mortgage (FRM) can be approximated by finding estimates of the terms Probability of Default (PD), Loss Given Default (LGD) and Exposure At Default

$$Loss^{FRM} = PD * LGD * EAD \tag{27}$$

This simple and straightforward model of course has its limitations. Although many times these terms are estimated separately, we can agree that these terms are not independent of each other and the dependence between them may even be nonlinear leaving the measurement of simple correlations to be insufficient. I first take this basic model as a basis to analyze the problem and point out some interesting observations that can made and could lead to more detailed modeling that better fits the problems that can be seen on the mortgage market.

Expanding the Basel II idea and presenting its limitations

I expand these terms with more detail by breaking it up further on cases of the Loan To Value (LTV) ratio using the Total Expectation Theorem:

$$\begin{aligned}
\mathbb{E}[Loss] &= \mathbb{P}(D)\mathbb{E}[Loss|D] \\
&= \mathbb{P}(D)\left(P(LTV \le 1|D)\mathbb{E}[Loss|D, LTV \le 1] + P(LTV > 1|D)\mathbb{E}[Loss|D, LTV > 1]\right)
\end{aligned}$$

Handling these situations separately is promising because the expected losses are far different in these two basic LTV situations, taking in consideration whether or not there is enough collateral.

To add further details in the model let's note that a default event triggers a substantial workout cost that is usually added to the losses of the bank unless the value of the collateral is enough to cover both the remaining balance and the workout costs. Taking this extra cost into consideration we could replace the splitting point of the LTV values to separate the two scenarios in which there is either enough collateral so that the bank only suffers the interest payments losses resulting in a similar situation to a prepayment or the other scenario of not having enough collateral to cover remaining balance and extra costs. I'll represent the workout cost (c_w) as a percentage of the collateral assuming that this could be accurately estimated initially. After the new split the equations look like:

$$\begin{aligned}\mathbb{E}[Loss] &= \mathbb{P}(D)P(LTV \leq 1-c_w|D)\mathbb{E}[Loss|D, LTV \leq 1-c_w] + \\ &\quad \mathbb{P}(D)P(LTV > 1-c_w|D)\mathbb{E}[Loss|D, LTV > 1-c_w] \end{aligned} \tag{28}$$

We still can't omit the $LTV \leq 1-c_w$ scenario from the equations and assume its value to be zero as we have to consider remaining interest payment losses instead of just loss on the principal itself and we can only have an estimate of possible workout costs which is definitely not a fixed value.

We can see that adding more detail to the model using this path of thought has its limitations because of using splitting points that will have big variances and so makes using the splits pointless. The problem may be that using equation (27) as a starting point and trying to further segment the probability space to give more detailed models we may approach the problem from the wrong starting point, namely starting with concentrating on the probability of default initially. We can also say that the resulting branches like the term $P(LTV > 1-c_w|D)$ is very counter intuitive and *we might miss a hidden cause and effect relationship*.

Taking a fresh start on segmenting the probability space

We could more easily find a good business interpretation of a reversed form like $P(D|LTV > 1-c_w)$ as the probability of a high LTV induced default where the borrower might have been able to continue payments but chose not to because the collateral is worth less than he owes the bank. By taking the LTV value to be the first branch we get the equation:

$$
\begin{aligned}
\mathbb{E}\left[Loss\right] \quad = \quad & \mathbb{P}(LTV \leq 1)P(D|LTV \leq 1)\mathbb{E}\left[Loss|LTV \leq 1, D\right] \\
& + \mathbb{P}(LTV \leq 1)\underbrace{P(\neg D|LTV \leq 1)\mathbb{E}\left[Loss|LTV \leq 1, \neg D\right]}_{=0} \\
& + \mathbb{P}(LTV > 1)P(D|LTV > 1)\mathbb{E}\left[Loss|LTV > 1, D\right] \\
& + \mathbb{P}(LTV > 1)\underbrace{P(\neg D|LTV > 1)\mathbb{E}\left[Loss|LTV > 1, \neg D\right]}_{=0} \\
= \quad & \underbrace{\mathbb{P}(LTV \leq 1)P(D|LTV \leq 1)\mathbb{E}\left[Loss|LTV \leq 1, D\right]}_{low LTV default loss} \\
& + \underbrace{\mathbb{P}(LTV > 1)P(D|LTV > 1)\mathbb{E}\left[Loss|LTV > 1, D\right]}_{high LTV default loss} \qquad (29)
\end{aligned}
$$

The first approach was based on segmenting the probability space from the bank's perspective to identify scenarios where there is enough collateral and where isn't. Reversing the questions and the conditions leads to a totally new perspective, namely segmenting the probability space from the customer's perspective to model his behavior and so identify scenarios when it is a rational decision to choose to default and refuse paying or when it is not a rational decision to default (and so it is not worth it) but the borrower has no other option. By using this modeling approach we don't omit the nonlinear dependence of the default probability on the LTV and the hidden cause and effect relationship if the $LTV > 1$ event occurs. I would expect $P(D|LTV \leq 1)$ to be less then $P(D|LTV > 1)$ and if empirical evidence would suggest that then the model should give decent results as this will break down defaults to two completely different cases that would now incorporate rational decisions. Results are presented in later chapters with measurements on data.

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