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ON EQUILIBRIA OF TETRAHEDRA

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ABSTRACT. A solid body, resting on a plane under gravitational force, must have at least one stable equilibrium point on its boundary (minimizing the distance from the center of mass) and one unstable equilibrium point (maximizing the same distance.) A body with a unique stable (resp. unstable) equilibrium is monostable (resp. mono-unstable.) If it has at least one of these properties it is monostatic; if both, mono-monostatic.

Conway and Guy [2] showed that a homogeneous polyhedron can be monostable, but that a homogeneous tetrahedron has at least two stable equilibrium points. The same idea has been used [7] to prove that a homogeneous tetrahedron has at least two unstable equilibria. Conway [3] also claimed that an inhomogeneous tetrahedron may be monostable. Here we give a formal proof of this statement, and show that all monostatic tetrahedra have exactly 4 equilibria. We also show that certain patterns of obtuse dihedral (resp. face) angles are equivalent to the existence of a monostable (resp. mono-unstable) weighting.

Our results imply that mono-monostatic tetrahedra do not exist. In contrast, we show that for any other legal number of faces, edges, and vertices there is a mono-monostatic polyhedron with that face vector.

On Equilibria of Tetrahedra

Gergő Almádi, Robert J. MacG. Dawson, Gábor Domokos, and Krisztina Regős

The curious mechanical properties of tetrahedral solids were first studied in 1967 when Aladár Heppes constructed a homogeneous tetrahedron [8] that could rest stably on only two of its faces. John Horton Conway then showed [3] that for a homogeneous tetrahedron, this number cannot be improved; i.e., it cannot be monostable.

A three-dimensional weighted convex polyhedron $\mathscr{P}$ with center of mass O supported by a fixed horizontal plane has three sorts of equilibria: stable (on a face), unstable (on a vertex), and saddle (on an edge). These correspond to local minima, maxima, and saddle points of the radial height function $r_{\mathscr{P},O} : S^2 \to \mathbb{R}^+$ describing the boundary of $\mathscr{P}$ as a distance measured from O. The global study of such equilibrium points, today associated with Morse theory, goes back (on smooth surfaces interpreted in a Cartesian coordinate system) to Arthur Cayley [1] in 1859. James Clerk Maxwell [9] noted a few years later that Euler's formula $f - e + v = 2$ describes the relationship between maxima, saddles, and minima; applied to the radial height function $r_{\mathscr{P},O}$, this links the numbers of those faces, edges, and vertices of the convex polyhedron $\mathscr{P}$ that carry equilibria.

Conway claimed that monostability is possible for weighted tetrahedra, and he asked whether monostability was possible for homogeneous simplices in higher dimensions. This was answered in a series of papers [3, 4, 6] by Dawson and Finbow, who also showed [5] that even some regular polytopes, appropriately weighted, can be monostable.

Here we focus on the weighted case. We will exhibit conditions on a (nonregular) tetrahedron $\mathscr{T}$ equivalent to the existence of a weighting $(\mathscr{T}, O)$ making it monostable. Surprisingly, a tetrahedron that meets this criterion can, with suitable weighting, be monostable on any face. We will also give a similar set of conditions for a tetrahedron to have a monounstable weighting.

Some bodies with interesting stability properties are very sensitive to variation in shape. The monomonostatic Gömböc, for instance, has a shape tolerance of about 0.1%, and one would be hard put to distinguish a genuine Gömböc from an impostor by visual inspection alone. Indeed, it has been shown [11] that smooth homogeneous monomonostatic bodies in $\mathbb{R}^3$ are, in a quantifiable sense, nearly spherical. In contrast, our criteria for tetrahedra are qualitative, involving only the obtuseness of certain face and dihedral angles.

We will show that every monostable tetrahedron is biunstable, that is, that it has equilibria on exactly two vertices. Neither Gömböc-like monomonostatic weighted tetrahedra nor monostable tetrahedra with three or four unstable equilibria exist. We will also show that while in general, the physics of tipping bodies in three or more dimensions is highly complicated, the tipping of a real-world monostable tetrahedron can be described with minimal calculation, given its shape and center of mass.

The polar dual of a polyhedron $\mathscr{P}$ is by definition the set $\mathscr{P}^* = \{x : x \cdot p \leq 1 \text{ for all } p \in \mathscr{P}\}$. If $\mathscr{P}$ is convex, then $\mathscr{P}^{**} = \mathscr{P}$. The polar dual of a tetrahedron is also a tetrahedron, and there is a natural pairing between the vertices of one and the faces of the other. The following result was proved in [7].

Proposition 1. *Let $(\mathscr{P}, O)$ be a weighted convex polyhedron with O at the origin. Then $\mathscr{P}$ has an equilibrium on a face if and only if $\mathscr{P}^*$ has an equilibrium on the corresponding vertex.*

Thus every monounstable weighted tetrahedron is bistable. We will exhibit explicit geometric conditions for a tetrahedron to have such a weighting; and we can see that (in contrast to the monostable case) if $(\mathscr{T}, O)$ is monounstable on a vertex A, then $\mathscr{T}$ cannot be weighted to be monounstable on any other vertex. Finally, we will show that the face vector $(f, e, v) = (4, 6, 4)$ that characterizes tetrahedra is unique among those of polyhedra, in that any other legal face vector has a representative polyhedron that may be weighted to make it monomonostatic.

Definitions of Equilibrium

Definition 2 ([7, 11]). Let $\mathscr{P}$ be a convex polyhedron, and let int $\mathscr{P}$ and bd $\mathscr{P}$ denote its interior and boundary. We select a point $O \in \text{int}\,\mathscr{P}$, which we shall think of as the center of mass. (We are not assuming $\mathscr{P}$ to have uniform density, so this implies no restriction on O other than its being an interior point.)

We say that $(\mathscr{P}, O)$ is in equilibrium on a face, edge, or vertex A if there exists $Q \in \text{relint}\,A$ (the relative interior of A) such that the plane perpendicular to $[O, Q]$ at Q supports $\mathscr{P}$. (Recall that the relative interior is defined in such a way that a singleton's relative interior is itself, though its interior is empty. Thus $\mathscr{P}$ may be in equilibrium on a vertex.) We call an equilibrium stable if Q is on the relative interior of a face, unstable if Q is a vertex, and hyperbolic

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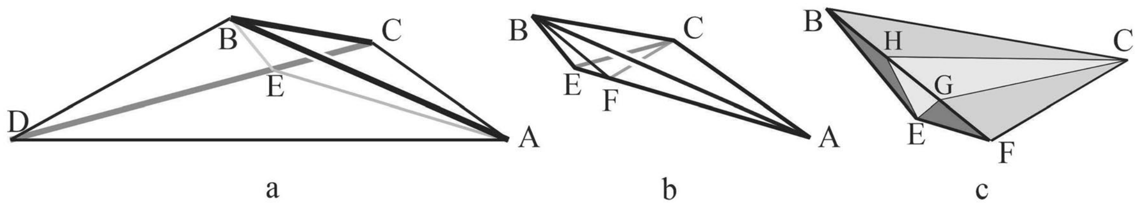

Figure 1. Loading regions for a tetrahedron with an obtuse path.

(saddle) otherwise, and we denote their respective numbers by S, U, H.

As noted above, Maxwell showed that

$$S - H + U = 2. \tag{1}$$

These equilibria correspond intuitively to positions in which a physical model of $(\mathscr{P}, O)$ balances on a horizontal surface. They also correspond to "pits," "peaks," and "passes" in the radial function of $\mathscr{P}$ with respect to O.

Definition 3 ([7]). We call a convex polyhedron $\mathscr{P}$ monostable if it has a unique stable equilibrium ($S = 1$; there is exactly one face on which it will rest) and monounstable if it has a unique unstable equilibrium ($U = 1$; there is exactly one vertex on which it can balance precariously). It is monostatic if $S = 1$, $U = 1$, or both; and monomonostatic if $S = U = 1$.

Results on Monostability

Henceforth we assume that $\mathscr{P}$ is a tetrahedron $\mathscr{T} = \boxtimes ABCD$ with face $\mathscr{A}$ opposite vertex A, etc. The next result gives a simple qualitative criterion for a tetrahedron to have a monostable weighting.

Obtuse dihedral angles play an important role in the stability of polyhedra: if a polyhedron can tip across an edge, then the dihedral angle at that edge must be obtuse. We say that a tetrahedron has an obtuse path A–B–C–D if it has three edges $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ with obtuse dihedral angles not all sharing a vertex.

Theorem 4. *Let $\mathscr{T}$ be a tetrahedron; then the following are equivalent:*

1. *$\mathscr{T}$ has an obtuse path;*
2. *there exists O such that $(\mathscr{T}, O)$ is monostable;*
3. *for every face F, there exists O_F such that $(\mathscr{T}, O_F)$ is monostable on F.*

Proof. (3) $\Rightarrow$ (2) trivially.

(2) $\Rightarrow$ (1): There must be enough obtuse dihedrals to let the tetrahedron roll from any face to the resting face. But no face of a tetrahedron can have three obtuse dihedrals, and no tetrahedron can have four obtuse dihedrals. Thus it must be possible to order the faces $(\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3, \mathscr{F}_4)$ with an obtuse dihedral angle between $\mathscr{F}_n$ and $\mathscr{F}_{n+1}$ for $n = (1, 2, 3)$ and no other obtuse dihedral angles. But the edges $\mathscr{F}_n \cap \mathscr{F}_{n+1}$ and $\mathscr{F}_{n+1} \cap \mathscr{F}_{n+2}$ have a point in common for $n = 1, 2$. Thus the obtuse dihedrals form a path of length 3, as desired.

Finally, we prove (1) $\Rightarrow$ (3). If our obtuse path is A–B–C–D, the obtuse edges connect the faces in the order $\mathscr{C}$–$\mathscr{D}$–$\mathscr{A}$–$\mathscr{B}$ (Figure 1). It suffices to show that for appropriate $O_{\mathscr{A}}$, the pair $(\mathscr{T}, O_{\mathscr{A}})$ is monostable on $\mathscr{A}$, and similarly for $\mathscr{B}$. Construct the plane perpendicular to $\mathscr{C} = \triangle ABD$ containing the edge $\overline{AB}$ shared with $\mathscr{D}$. This cuts $\overline{CD}$ at a point E (Figure 1a). The tetrahedron $\boxtimes ABCE$ has obtuse edges $\overline{BC}$ and $\overline{EC}$. If $O \in \operatorname{int} \boxtimes ABCE$, then $(\mathscr{T}, O)$ has no stable equilibrium on $\mathscr{C}$. Next, construct the plane perpendicular to $\mathscr{D} = \triangle ABC$ containing $\overline{BC}$. This cuts $\overline{AE}$ at F, and $\overline{CE}$ is an obtuse edge of the tetrahedron $\boxtimes BCEF$ (Figure 1b). If $O \in \operatorname{int} \boxtimes BCEF$, then $(\mathscr{T}, O)$ has no stable equilibrium on $\mathscr{C}$ or $\mathscr{D}$.

If we repeat this with a plane perpendicular to $\mathscr{A}$ containing $\overline{CE}$, it meets $\overline{BF}$ at G; and if $O_{\mathscr{B}} \in \operatorname{int} \boxtimes CEFG$, then $(\mathscr{T}, O_{\mathscr{B}})$ has no stable equilibrium on $\mathscr{A}$, $\mathscr{C}$, or $\mathscr{D}$, so it is monostable on $\mathscr{B}$ (Figure 1c). Similarly, the plane containing the same edge but perpendicular to $\mathscr{B}$ meets $\overline{BF}$ at H; and if $O_{\mathscr{A}} \in \operatorname{int} \boxtimes BCEH$, then $(\mathscr{T}, O_{\mathscr{A}})$ has no stable equilibrium on $\mathscr{B}$, $\mathscr{C}$, or $\mathscr{D}$ and is monostable on $\mathscr{A}$. $\square$

Such tetrahedra exist. For instance, it can be easily verified that if

$$(A, B, C, D) = \big((-20, 0, 0), (-1, -1, 10), (1, 1, 10), (20, 0, 0)\big)$$

are the vertices of a tetrahedron, then A–B–C–D is an obtuse path, and on face $\mathscr{D}$, the tetrahedron is monostable with its center of mass located at $O = (-2, 0, 9)$.

Remark 5. We have shown that suitably weighted, a tetrahedron with an obtuse path is stable only on one face. We have not yet shown how it gets there. (As the bartender says at closing time, "You don't have to go home, ladies and gentlemen, but you can't stay here.") In fact, without some way to dissipate energy, the tetrahedron will never settle onto any face but will instead bounce forever.[1]

When tipping over an edge (Figure 2a), a polyhedron has only one degree of freedom, and how it dissipates its energy does not affect where it ends up, as long as it does so effectively. However, in some cases, the body may tip not

[1]Fans of opera, or at least of operatic trivia, will recall the story of Eva Turner, in the role of Tosca, throwing herself from the battlements onto an over-resilient trampoline placed there by the stagehands and making several unplanned curtain calls.

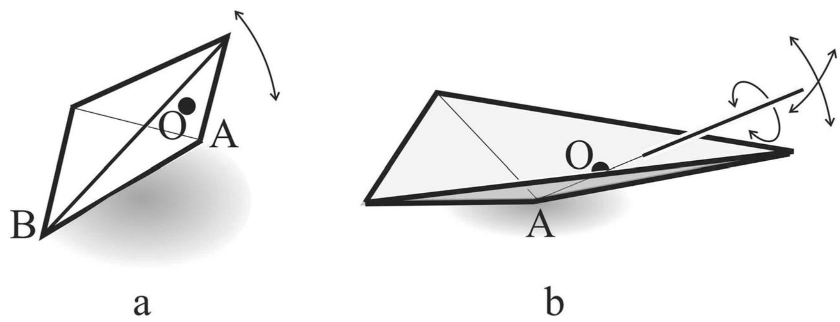

Figure 2. A tetrahedron that rolls without slipping can have one or three degrees of freedom

onto an edge (from where it must continue to the next face) but onto a vertex. Should this happen, the body temporarily has not two but three degrees of freedom: the center of mass O moves on a sphere about A, and the body can also rotate about the axis AO (Figure 2b). We thus need to take into account torque, moment of inertia, and the position of O relative to the edge on which the tetrahedron lands. The problem might seem intractable.

Fortunately, these difficulties never arise if $(\mathscr{T}, O)$ is a monostably weighted tetrahedron. In this case, as shown above, if a face has two obtuse dihedrals, the center of mass is positioned so that $\mathscr{T}$ will tip onto that face across at least one of those edges. Each nonequilibrium face thus has a unique exit; and provided that we assume landings to be reasonably inelastic, the exact path to stable equilibrium may be found easily knowing only $(\mathscr{T}, O)$ and the starting face.

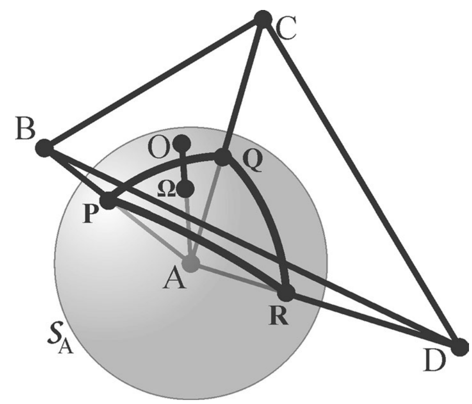

Figure 3. The geometry of a vertex is the geometry of a sphere

Some Spherical Geometry

If we consider the intersection of $\mathscr{T}$ with a small sphere $\mathscr{S}_A$ centered at some vertex A, we see that the geometry of polyhedral vertices is just that of the sphere. For a weighted tetrahedron $(\mathscr{T}, O)$, let P, Q, R, Ω be the respective intersections of $\mathscr{S}_A$ with $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, and $\overline{AO}$ (see Figure 3). Then (for instance) the face angle $\angle BAC$ corresponds to the arc $\overline{PQ}$ on $\mathscr{S}_A$, while the dihedral angle between faces $\triangle ABC$ and $\triangle ACD$ corresponds to the spherical angle $\angle PQR$. In each case, the radian measures are equal. We call a spherical segment short if its measure is less than $\pi/2$; otherwise, it is said to be long; angles, as usual, are "acute" or "obtuse."

The following result characterizes unstable equilibria.

Lemma 6. *For any vertex A of a weighted tetrahedron $(\mathscr{T}, O)$:*

1. *$(\mathscr{T}, O)$ has an (unstable) equilibrium on A if and only if the angles $\angle BAO$, $\angle CAO$, and $\angle DAO$ are all acute, equivalently, if and only if all of the spherical arcs $\overline{P\Omega}$, $\overline{Q\Omega}$, and $\overline{R\Omega}$ are short.*
2. *$(\mathscr{T}, O)$ has an equilibrium on A for every $O \in \operatorname{int} \mathscr{T}$ if none of the face angles $\angle BAC$, $\angle CAD$, $\angle DAB$ are obtuse, equivalently, if none of the arcs $\overline{PQ}$, $\overline{QR}$, and $\overline{RP}$ are long.*

Proof. Let Π be the plane normal to $\overline{OA}$ at A. Then $\mathscr{T}$ has an equilibrium on A if and only if B, C, D all lie on the same side of Π as O. This is true for every O interior to $\mathscr{T}$ if and only if $\angle BAC$, $\angle CAD$, $\angle DAB$ are all acute or right.

We can also characterize stable equilibria in this way, though we have local configurations at three vertices to consider. The following result is obvious:

Lemma 7. 1. *$(\mathscr{T}, O)$ has a stable equilibrium on $\triangle ABC$ if and only if the dihedral angles between (on the one hand) $\triangle ABC$ and (on the other hand) $\triangle ABO$, $\triangle AOC$, and $\triangle OBC$ are all acute.*

2. *$(\mathscr{T}, O)$ has a stable equilibrium on $\triangle ABC$ for every $O \in \operatorname{int} \mathscr{T}$ if and only if none of the dihedral angles between (on the one hand) $\triangle ABC$ and (on the other hand) $\triangle ABD$, $\triangle ADC$, and $\triangle DBC$ are obtuse.*
3. *If (for instance) the dihedral angle between $\triangle ABC$ and $\triangle ABO$ is obtuse, then (on $\mathscr{S}_A$) the angle $\angle QP\Omega$ is obtuse, as is the corresponding angle on the sphere $\mathscr{S}_B$.*

Now that we have seen the significance of spherical geometry in this problem, let us establish a few facts from mathematical folklore.

Lemma 8. 1. *A spherical triangle with only acute angles has only short edges.*
2. *A spherical triangle with exactly one acute angle has exactly one short edge, which is opposite the acute angle.*
3. *A spherical triangle with three long edges has three obtuse angles.*
4. *A spherical triangle with exactly one long edge has exactly one obtuse angle, opposite the long edge.*
5. *A spherical triangle with only short edges has at most one obtuse angle.*

Proof. 1. If $\triangle ABC$ has only acute angles, then $\cos A$, $\cos B$, and $\cos C$ are all positive. Then if (for instance) edge BC has radian length a, one of the spherical cosine laws gives us that

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C} > 0,$$

whence $a < \pi/2$. The proofs for b and c are similar.

2. If A is acute, let A' be the antipodal point: the colunar triangle $\triangle A'BC$ satisfies the conditions of (1).

3. If $\triangle ABC$ has only long edges, then $\cos a$, $\cos b$, and $\cos c$ are all negative; the other spherical cosine law gives

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} < 0,$$

and A is obtuse. The proofs for B and C are similar.

4. This again follows from (3) by consideration of the colunar triangle.

5. In this case, $\cos a$, $\cos b$, and $\cos c$ are all positive. If A is obtuse, then $\cos a < \cos b \cos c$, whence a, opposite A, must be the strictly longest edge. $\square$

We can, however, construct spherical triangles with exactly one obtuse angle and zero or two long edges. We can also construct a spherical triangle with three obtuse angles and only two long edges. These results are summarized in Table 1.

Results on Instability

We begin by ruling out the possibility of a "weighted tetrahedral Gömböc" and in fact prove more.

Theorem 9. *No tetrahedron has a monostable weighting and also a monounstable weighting, even with different centers of mass.*

Proof. As observed above, every monostable weighted tetrahedron $(\mathscr{T}, O)$ has two vertices B, C that each have two obtuse dihedrals. Table 1 shows that each of these two must have two obtuse face angles; but a tetrahedron cannot have more than four obtuse face angles in total, so the other two vertices, A, D, have only acute face angles. By Lemma 6, $(\mathscr{T}, O')$ has equilibria on those vertices for every $O' \in \operatorname{int} \mathscr{T}$. $\square$

Table 1. Possible combinations of long/obtuse elements in spherical triangles and polyhedral vertices

		Obtuse angles (dihedrals)			
Long		0	1	2	3
edges	0	√	√	x	x
(obtuse	1	x	√	x	x
face	2	x	√	√	√
angles)	3	x	x	x	√

For a specific weighting, we can say more.

Theorem 10. *If a weighted tetrahedron $(\mathscr{T}, O)$ is monostable, then it has unstable equilibria on exactly two vertices and a saddle equilibrium on exactly one edge.*

Proof. By Theorem 4, we may assume that $(\mathscr{T}, O)$ has an obtuse path A–B–C–D and an equilibrium on either $\triangle DBC$ or $\triangle ACD$. Since the dihedrals on $\overline{BC}$ and $\overline{CD}$ are obtuse, the dihedral on $\overline{BD}$ must be acute. However, by hypothesis, the other two dihedrals at B are obtuse.

The local geometry at B thus corresponds to a spherical triangle $\triangle \delta\gamma\alpha$ with an acute angle at δ and obtuse angles at γ and α. Then (Lemma 8) the edges $\overline{\delta\gamma}$ and $\overline{\delta\alpha}$ are long, and $\overline{\alpha\gamma}$ is short. Let E be polar to the great circle through the points δ and α; it lies (Figure 4) on the great circle polar to δ, which meets $\overline{\delta\gamma}$ at F and $\overline{\delta\alpha}$ at G.

But by assumption, $(\mathscr{T}, O)$ has no equilibrium on $\triangle DBA$, so $\angle\delta\alpha\Omega$ is obtuse. Thus Ω lies on the far side of $\overline{\alpha E}$, and a fortiori on the far side of $\overline{GE}$, from D. Thus $\overline{\delta\Omega}$ is long, $\angle DBO$ is obtuse, and $(\mathscr{T}, O)$ has no equilibrium on B. A similar argument (using the lack of equilibrium on $\triangle ABC$) shows that $(\mathscr{T}, O)$ has no equilibrium on C.

The unique saddle equilibrium follows from Maxwell's formula (1). $\square$

Using polar duality, we also get the following corollary.

Corollary 11. *If a weighted tetrahedron $(\mathscr{T}, O)$ is monounstable, then it has a stable equilibrium on exactly two faces, and a saddle equilibrium on exactly one edge.*

We can now prove a result analogous to Theorem 4 for monounstable tetrahedra that does not appear to follow from that theorem via polar duality. Define an obtuse cycle to be a cycle of edges A–B–C–D–A on a tetrahedron such that the face angles $\angle ABC$, $\angle BCD$, and $\angle CDA$ are all obtuse.

Theorem 12. *A tetrahedron $\mathscr{T}$ has an obtuse cycle if and only if for some O, the pair $(\mathscr{T}, O)$ is monounstable.*

Proof. Suppose that A–B–C–D–A is an obtuse cycle and $P \in \overline{BC}$. Then $\angle ABP = \angle ABC$ is obtuse. By the same

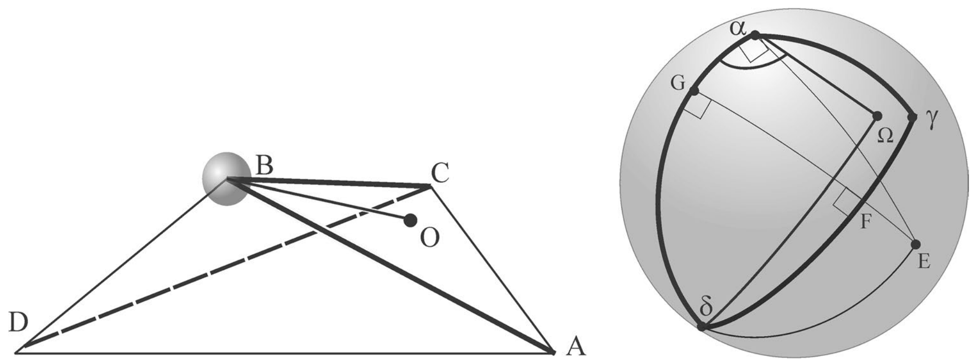

Figure 4. The configuration at a vertex with two obtuse dihedral angles.

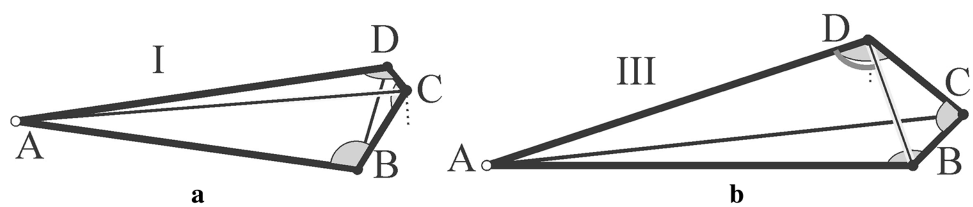

Figure 5. Tetrahedra with obtuse cycles.

argument, the angle $\angle DCP$ is obtuse. Moreover, since $\angle ADC$ is obtuse, so is $\angle ADP$ for $P \in$ relint $\overline{BC}$ close enough to C. Now, P is on the boundary of $\mathscr{T}$, but if we let O be an interior point close enough to P, then the angles $\angle ABO$, $\angle DCO$, and $\angle ADO$ will still be obtuse, and $(\mathscr{T}, O)$ will have no equilibrium on B, C, or D.

We now show that monoinstability requires the existence of an obtuse cycle. Since a triangle has at most one obtuse angle, a tetrahedron has at most four obtuse face angles; and to be monounstable, it must have one (or more) obtuse face angles at each of the three vertices without an equilibrium. The obtuse face angles can thus be partitioned among the vertices in only three ways: $\{0, 1, 1, 1\}$, $\{0, 1, 1, 2\}$, and $\{1, 1, 1, 1\}$.

We will represent a vertex with m obtuse face angles and n obtuse dihedrals by the ordered pair $[m, n]$. Since every obtuse dihedral has two ends, the sum of n over the vertices is even; and we can only use the pairs $[m, n]$ found in Table 1. The only possibilities for a monounstable tetrahedron are as follows:

I. $\{[0, 1], [1, 1], [1, 1], [1, 1]\}$;
II. $\{[0, 0], [1, 1], [1, 1], [2, 2]\}$;
III. $\{[0, 1], [1, 1], [1, 1], [2, 1]\}$;
IV. $\{[0, 1], [1, 1], [1, 1], [2, 3]\}$;
V. $\{[1, 1], [1, 1], [1, 1], [1, 1]\}$.

Possibility I, which has only three obtuse face angles, is realizable, for instance by a tetrahedron with vertices

$$\{(-10, 0, 0), (0, 2, 0), (0, -2, 0), (1, 0, 1)\}$$

(Figure 5a). Let A be the $[0, 1]$ vertex, and $\overline{AC}$ its obtuse dihedral. Then $\overline{BD}$ is also an obtuse dihedral; the face angles $\angle ABC$, $\angle BCD$, and $\angle CDA$ are obtuse; and A–B–C–D–A is an obtuse cycle.

Possibility II cannot occur. Let D be the $[2, 2]$ vertex, with obtuse angles $\angle ADB$ and $\angle ADC$. Then the dihedrals on $\overline{DC}$ and $\overline{DB}$ are obtuse, B and C are the vertices of type $(1, 1)$, and the angles $\angle ABC$ and $\angle ACB$ are both obtuse, which is impossible.

Possibility III can occur. Let A be the $[0, 1]$ vertex, D the $[2, 1]$ vertex. The tetrahedron has two obtuse dihedrals without a common endpoint. If they were $\overline{AD}$ and $\overline{BC}$, then one of the angles $\angle ADB$, $\angle ACD$ would be obtuse (without loss of generality, $\angle ADB$). But the angles opposite $\overline{BC}$, that is, $\angle ABD$ and $\angle ACD$, are also obtuse; so $\triangle ABD$ would have two obtuse angles, which is impossible.

However, if (without loss of generality) the obtuse dihedrals are $\overline{AC}$ and $\overline{BD}$, we can construct examples, for instance

$$(A, B, C, D) = ((-10, 0, 0), (2, 0, 0), (3, 2, 0), (0, 4, 1))$$

(Figure 5b). Here A–B–C–D–A is the obtuse cycle.

Possibility IV cannot occur. Let A be the $[2, 3]$ vertex; then the dihedrals on $\overline{AB}$, $\overline{AC}$, and $\overline{AD}$ are all obtuse. Without loss of generality let C and D be the vertices of type $[1, 1]$; as before, the angles $\angle CDB$ and $\angle BCD$ are both obtuse.

Possibility V would require the tetrahedron to have two disjoint obtuse dihedrals (without loss of generality, $\overline{AC}$, $\overline{BD}$) opposite the four obtuse angles; but then the skew quadrilateral $\bowtie ABCD$ would have angles summing to more than 2π, which is impossible. $\square$

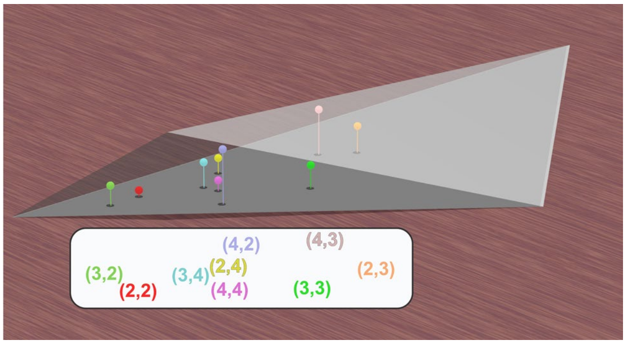

Figure 6. A tetrahedron that can have two to four stable equilibria and two to four unstable equilibria, depending on the choice of center.

Table 2. Possible combinations of equilibria

		Unstable equilibria			
		1	2	3	4
Stable equilibria	1	x	√	x	x
	2	√	√	√	√
	3	x	√	√	√
	4	x	√	√	√

Remark 13. By Theorems 4 and 12 and Maxwell's formula (1), the following are equivalent for any weighted tetrahedron $(\mathscr{T}, O)$:

1. $(\mathscr{T}, O)$ is monostatic;
2. $(\mathscr{T}, O)$ has exactly one saddle equilibrium;
3. $(\mathscr{T}, O)$ has exactly four equilibria.

Examples of weighted tetrahedra with every combination of two to four stable equilibria and two to four unstable equilibria are given in [7]. Indeed, there exists a single tetrahedron that exhibits all nine of these combinations for appropriate choices of center (Figure 6).[2] From this example, it follows that the monostable and monounstable cases are the only ones that can be characterized by the shape of the tetrahedron.

Remark 14. If a weighted tetrahedron $(\mathscr{T}, O)$ is monounstable with an equilibrium at vertex A, then A has no obtuse face angles. It follows that $(\mathscr{T}, O')$ has an equilibrium on A for all $O' \in \operatorname{int} \mathscr{T}$, and thus (in contrast to the situation in Theorem 4) $\mathscr{T}$ cannot be weighted to be monounstable on any other vertex. Results like this show that (despite our use of polar duality and the symmetry of Table 2) there is no simple duality between stable and unstable equilibria.

Other Polyhedra

We have seen that no tetrahedron can be monomonostatic, even when weighted. What about other classes of polyhedra? A vector $(f, e, v) \in \mathbb{N}^3$ is the face vector of some nondegenerate polyhedron if and only if

- $f \geq \frac{v}{2} + 2$,
- $v \geq \frac{f}{2} + 2$,
- $e = f + v - 2$.

We shall call such a vector legal. We note that equality is obtained in the first expression only when all vertices have degree 3, and the second only when all faces are triangular.

Theorem 15. *Every legal vector except for* $(4, 6, 4)$ *is the face vector of a monomonostatic weighted polyhedron.*

Proof. Let $\mathscr{P}$ be a weighted polyhedron with at least one nontriangular face. We claim that some vertex V of $\mathscr{P}$ is included in one, two, or three nontriangular faces. For suppose otherwise. Then intersecting the half-spaces bounded by supporting planes on these faces and containing P, we get a convex polyhedron $\mathscr{Q}$ with at least four edges at every vertex and at least four edges on every face. Then $e \geq 2v$, $e \geq 2f$, and so for the Euler characteristic we have $\chi(\mathscr{Q}) \leq 0$, an impossibility.

Let V be such a vertex, let F be a nontriangular face including V, and let $\delta > 0$. We will construct a new polyhedron $\mathscr{P}'$ that shares every vertex of $\mathscr{P}$ except that V is replaced by a new vertex V'. Let G be the intersection of the affine hulls of the other nontriangular faces (if any) of $\mathscr{P}$ at V. It is an affine subspace of dimension at least 1. Let H be the open half-space bounded by aff F that contains int $\mathscr{P}$. Then take $V' \in G \cap H \cap B_\delta(V)$. Clearly, at least for small δ, $\mathscr{P}'$ has the same number of vertices as $\mathscr{P}$, and one more face. Moreover, by taking δ small enough, we can change the orientations of edges and faces by an angle less than any desired $\epsilon > 0$. We shall refer to this below as face bending.

Let $(\mathscr{P}, O)$ be a weighted polyhedron; we assume O to be in general position with respect to all edges and face diagonals. For small enough δ, the following are true:

- $O \in \operatorname{int} \mathscr{P}'$;
- $(\mathscr{P}', O)$ has an equilibrium on a vertex X if and only if $(\mathscr{P}, O)$ has an equilibrium on the corresponding vertex;

[2] The vertices are (0, 0, 0), (100 000, 0, 0), (50 000, 41 429, 0), and (13 549, 13 544, 11 223). The centers are $M_{22} = (15\,884, 5116, 835)$, $M_{23} = (46\,670, 11\,911, 3061)$, $M_{24} = (28\,497, 5544, 2041)$, $M_{32} = (11\,400, 7243, 2597)$, $M_{33} = (33\,447, 17\,389, 3061)$, $M_{34} = (23\,866, 8138, 3339)$, $M_{42} = (21\,845, 14\,097, 7142)$, $M_{43} = (42\,514, 9100, 6122)$, and $M_{44} = (24\,407, 10\,239, 1391)$.

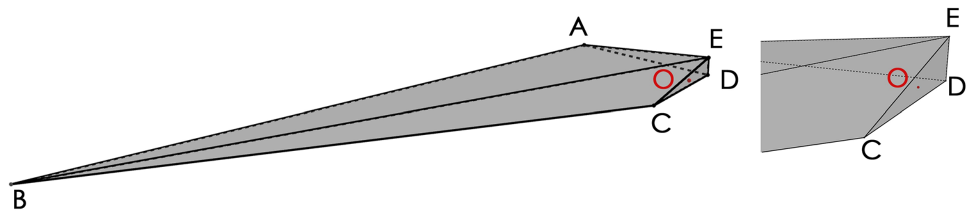

Figure 7. A monomonostatic polyhedron with face vector (5, 8, 5). Vertices are (0, 0, 0), (10 000, 0, 0), (10 000, 2890, 0), (11 216, 1008, 0), and (11 216, 968, 280); the center of mass is (10 790, 643, 84).

- $(\mathscr{P}, O)$ has an equilibrium on any face other than F if and only if $(\mathscr{P}', O)$ has an equilibrium on the corresponding face;
- $(\mathscr{P}, O)$ has an equilibrium on F if and only if $(\mathscr{P}', O)$ has an equilibrium on F' or T (this requires the foot of the perpendicular from O to F not to lie on the face diagonal that becomes an edge of F');
- $(\mathscr{P}', O)$ cannot have equilibria on both F' and T.

It follows that if there exists a monomonostatic polyhedron that has v vertices and f faces not all triangles, then there exists such a polyhedron with v vertices and $f + 1$ faces. As shown in [7], by polar duality there then also exists one with f vertices and v faces.

We conclude the proof using induction. First, we note that there exists a monomonostatic polyhedron with face vector (5, 8, 5) (combinatorially equivalent to a square pyramid); see Figure 7.

Assume, as our inductive hypothesis, that the claim holds for every legal vector with $v \leq 2n$. Then in particular, it holds for $(f, e, v) = (2n + 1, 3n + 2, n + 3)$ and $(f, e, v) = (2n + 2, 3n + 3, n + 3)$. By polar duality, it also holds for $(f, e, v) = (n + 3, 3n + 2, 2n + 1)$ and $(f, e, v) = (n + 3, 3n + 3, 2n + 2)$, both of which minimize f for the given v. Face bending then shows that the claim holds for all legal vectors with $v \leq 2n + 2$, hence by induction for all legal vectors. □

Except in a few cases, the vector (f, e, v) does not determine the combinatorial class of a polyhedron. We conjecture that in fact, every combinatorial class of polyhedra except the tetrahedra contains elements that admit a monomonostatic weighting.

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References

[1] A. Cayley. On contour and slope lines. *Phil. Mag.* XVIII (1859), 264–268.

[2] J. H. Conway and R. K. Guy. Stability of polyhedra. *SIAM Rev.* 11 (1969), 78–82.

[3] R. Dawson. Monostatic simplexes. *Amer. Math. Monthly* 92 (1985), 541–546.

[4] R. Dawson, W. Finbow, and P. Mak. Monostatic simplexes II. *Geom. Dedicata* 70 (1998), 209–219.

[5] R. Dawson and W. Finbow. What shape is a loaded die? *Mathematical Intelligencer* 21:1 (1999), 32–37.

[6] R. Dawson and W. Finbow. Monostatic simplexes III. *Geom. Dedicata* 84 (2001), 101–113.

[7] G. Domokos, F. Kovács, Z. Lángi, K. Regős, and P. T. Varga. Balancing polyhedra. *Ars Mathematica Contemporanea* 19:1 (2020), 95–124.

[8] A. Heppes. A double tipping tetrahedron. *SIAM Rev* 9:3 (1967), 599–600.

[9] J. C. Maxwell. On hills and dales. *Phil. Mag.* XXIX (1870) 233–240.

[10] J. Milnor. *Morse Theory*. Princeton University Press, 1963.

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[11] P. L. Várkonyi and G. Domokos. Mono-monostatic bodies: the answer to Arnold's question. *Mathematical Intelligencer* 28 (2006), 34–38.